Topics in Operator Theory. Winter 2022 Lecture Notes

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Course outlines

- Unbounded operators: general facts and definitions, first examples.

  (Domain of definition, Extensions, polarization identity, graphs, closed operators, adjoint operator, symmetric and self-adjoint operators, graph technique, self-adjointness of A^*A, $i\frac{d}{dx}$ on bounded interval, essentially self-adjoint operators, operator of the Dirichlet problem on the interval, harmonic oscillator, operator of the multiplication by a measurable function)

- Friedrichs extension

- Resolvent, Resolvent of a self-adjoint operator, Herglotz theorem, Spectral theorem (Wintner's proof following Berezin and Shubin).

- Kato-Rellich theorem and and self-adjointness of Schroedinger operator (hydrogen, etc).

- Theory of self-adjoint extensions, deficiency indices, von Neumann formulae

Two additional topics taken from Winter 2015 lecture notes

1. Pseudo-Laplacians in R^d.
2. Comparison of self-adjoint extensions: Krein formula for resolvents.

Appendix

1. Compact self-adjoint operators

Miscellanea

1. Zorn Lemma
2. UBP, compact operators
3. Vishik-Lax-Milgram Theorem
4. A Lemma on generalized functions (needed for the proof of Sears criterion in Berezin-Shubin).

1 General facts about unbounded operators (Dry Desert)

LECTURE 1

1.1 Domains of definition, extensions, polarization identity

H - a SEPARABLE (avoiding Zorn stuff, just a technical simplification) COMPLEX (real theory is a bit special, although also interesting) Hilbert space.

$(\cdot, \cdot)$ - linear w. r. t. THE FIRST argument, anti-linear w. r. t. THE SECOND argument.

Physicists and some authors use the opposite agreement.

In quantum mechanics textbooks: $(\cdot, \cdot) =< \cdot, \cdot >$, ($<|=$ bra; $|> =$ (c)ket vectors. Ket-vectors = vectors; bra-vectors = linear functionals.

Riesz theorem: $l(x) = (x, v_l)$ for a linear functional; for physicists and authors close to physics $l(x) = (v_l, x) =< v_l|x >$.

Let

$$A : H \to H$$

be a linear operator with **DENSE** $D(A)$ (domain (of definition)).

Remark: For all operators in this course $D(A)$ is always a linear subspace of H and is always dense (i. e. $\overline{D(A)} = H$).

Clearly, if A is bounded then it can be extended to the whole H. So, typically, in this course A is unbounded.

Example: $\mathbb{I} := \{x \in \mathbb{R} : 0 \leq x \leq 1\}$; H=$L_2(\mathbb{I})$; $A = \frac{d}{dx}$; $D(A) = C^1(\mathbb{I})$.

Exercise 1: Prove that A is unbounded.
Hint: consider normalized t^n.

Example-advertisement: Schroedinger operator $-\Delta + q(x)$ (say, in $L_2(\mathbb{R}^3)$). Theory is as large and deep as mathematics as a whole.

Work with unbounded operators requires a lot of care because of troubles with domains.

$$D(A + B) = D(A) \cap D(B)$$

$$D(AB) = \{x \in D(B) : Bx \in D(A)\}$$

Definition 1. *Extension:* $A \subseteq B$ *iff* $D(A) \subseteq D(B)$ *and* $B|_{D(A)} = A$.

Be careful: $A(B + C) \neq AB + AC$ (example: $C = -B$)
Exercise 2: Prove

$$AB + AC \subseteq A(B + C)$$

but

$$(B + C)A = BA + CA$$

Let $x, y \in D(A)$. Then one has **polarization identity**:

$$\boxed{(Ax, y) = \frac{1}{4} \sum_{\epsilon = \pm 1, \pm i} \epsilon \left(A(x + \epsilon y), (x + \epsilon y) \right)}$$

(Remark: mind that for us $(\cdot, \cdot)$ is anti-linear w. r. t. **the second** argument. For physicist's choice the r. h. s. should be changed to conjugate - aesthetically worse!! - and that is one of the most important reasons to make mathematician's choice.)

Exercise: Prove via direct calculation. Once in your life you have to do that (similarly to the Jacobi identity for $[a, [b, c]]$ with $[a, b] = ab - ba$!)

Meaning: sesquilinear form is defined by quadratic one.

Exercise 3. $(\forall x \in D(A)\ (Ax, x) \in \mathbb{R}) \implies (\forall x, y \in D(A)\ (Ax, y) = (x, Ay))$ (symmetry).

1.2 Graphs, Closed operators

Graph of A:

$$\Gamma(A) := \{[x, Ax] \in H \times H; x \in D(A)\}$$

Let $x, y \in D(A)$. Define

$$\langle\!\langle x, y \rangle\!\rangle := (x, y) + (Ax, Ay)$$

Graph norm:

$$|||x||| = \langle\!\langle x, y \rangle\!\rangle^{1/2}$$

The following three statements are equivalent:

- $\Gamma(A)$ is closed in $H \times H$
- $D(A) \ni x_n \to x_0;\ Ax_n \to y_0 \implies x_0 \in D(A)$ and $Ax_0 = y_0$
- $D(A)$ is Hilbert space w. r. t. graph norm

If any of these three facts holds true then operator A is called closed.

Definition 2. *A is closable if the closure, $\overline{\Gamma(A)}$, of $\Gamma(A)$ is the graph of a linear operator.*

(i. e. $[x, y_1], [x, y_2] \in \overline{\Gamma(A)} \implies y_1 = y_2$; $\overline{\Gamma(A)}$ (as well as $\Gamma(A)$) is a linear space; therefore, this is equivalent to $[0, y] \in \overline{\Gamma(A)} \implies y = 0$ - prove!)

Example. $L_2(\mathbb{I})$, $A : f \mapsto f(1) \in L_2(\mathbb{I})$; $D(A) = C(\mathbb{I})$. Non-closable. $[t^n, 1] \in \Gamma(A)$; $t^n \to 0$ in $L_2(\mathbb{I})$. $[0, 1] \in \overline{\Gamma(A)}$.

Exercise 4. $D \subset H$ linear, dense. $F : D \to \mathbb{C}$ - unbounded linear functional;

$$Ax := F(x)h_0$$

with some given non-zero $h_0 \in H$. Prove that A with $D(A) = D$ is non-closable.

Solution: $x_n \in D$; $x_n \to 0$, $|F(x_n)| \geq \epsilon > 0$. $A(x_n/F(x_n)) = const = h_0$; $x_n/F(x_n) \to 0$. So $(0, h_0) \in \overline{\Gamma(A)}$.

Definition 3. *A - symmetric if $\forall x, y \in D(A)$ $(Ax, y) = (x, Ay)$.*

Proposition 1. *Symmetric operators are closable.*

We have to prove that

$$[x, y_1], [x, y_2] \in \overline{\Gamma(A)} \implies y_1 = y_2$$

Let

$$(x_n, Ax_n) \to (x, y_1)$$
$$(\tilde{x}_n, A\tilde{x}_n) \to (x, y_2)$$
$$x_n, \tilde{x}_n \in D(A)$$

Let $z \in D(A)$

$$(Ax_n, z) = (x_n, Az)$$

and, therefore ($n \to \infty$)),

$$(y_1, z) = (x, Az)\,.$$
$$(A\tilde{x}_n, z) = (\tilde{x}_n, Az)$$

and, therefore

$$(y_2, z) = (x, Az)$$

Thus

$$(y_1, z) = (y_2, z)$$

and ($D(A)$ is dense) $y_1 = y_2$.

Definition 4. *Let A be closable. Closed operator with graph $\overline{\Gamma(A)}$ is called the closure of A and is denoted by $\overline{A}$.*

Excercise 5. Closure $\overline{A}$ of symmetric operator A is symmetric.

Reminder: Closed graph theorem: closed linear operators defined on the whole (Banach) space are continuous (= bounded).

Proposition 2. *Helinger-Toeplitz Theorem.*
A is symmetric and $D(A) = H \implies A$ is bounded.

Proof. Obvious: A - closable, and, therefore $A = \overline{A}$, so A is bounded.

Corollary: A is symmetric and unbounded $\implies D(A) \neq H$.

END OF LECTURE 1

LECTURE 2

Philosophical question: Why we are working with closed operators?

Philosophical answer: Non-closed operators have non interesting spectrum which gives no information about the operator.

Definition 5. *Complement to the spectrum:*

$$(\mathrm{Spec}\,(A))^C = \{z \in \mathbb{C} : (A - zI)^{-1} \ \ \textit{exists} \ \ \textit{and} \ \ \textit{is} \ \ \textit{bounded}\}$$

(In particular, $A - zI : D(A) \to H$ is one to one.)

Proposition 3. $\mathrm{Spec}\,(A) \neq \mathbb{C} \implies A$ *is closed.*

Proof. Let $(A - zI)^{-1} : H \to D(A)$ be bounded and one to one for some z. Let $[x_n, Ax_n] \to [x_0, y_0]$ for $x_n \in D(A)$. we have to show that $x_0 \in D(A)$ and $Ax_0 = y_0$.

$$(A - z)x_n \to y_0 - zx_0$$

$(A - z)^{-1}$ is continuous, therefore,

$$x_n \to (A - z)^{-1}(y_0 - zx_0)$$

Thus,

$$x_0 = (A - z)^{-1}(y_0 - zx_0) \in D(A)$$

On the other hand

$$(A - z)x_0 = y_0 - zx_0$$

and, therefore,

$$Ax_0 = y_0$$

1.3 Adjoint Operator, Self-adjoint operators

Let $y \in H$. Consider the linear functional $D(A) \to \mathbb{C}$

$$x \mapsto (Ax, y)\,.$$

There is no reason to expect it is continuous unless A is a continuous operator. But for some y it may happen.

Example. $L_2(\mathbb{I})$, $A = \frac{d}{dt}$, $D(A) = C_0^1(\mathbb{I})$

$$\int_0^1 \left(\frac{d}{dt}x\right)\bar{y} = -\int_0^1 x\left(\frac{d}{dt}\bar{y}\right)$$

if $y \in C^1(\mathbb{I})$ and

$$|(Ax, y)| \leq c||x; L_2||$$

If this is the case (i. e. $x \mapsto (Ax, y)$ is continuous) then Riesz theorem implies

$$(Ax, y) = (x, z)$$

for some $z \in H$.

Definition 6. *If for some y the functional $D(A) \ni x \mapsto (Ax, y)$ is continuous and $(Ax, y) = (x, z)$ for all $x \in D(A)$, then they say that $y \in D(A^*)$ and $A^*y = z$.*

Thus,

$$(Ax, y) = (x, A^*y)$$

for $x \in D(A)$ and $y \in D(A^*)$.

Clearly, $D(A^*) \ni \{0\}$ and,therefore, is never empty.

Philosophical remark. Complete description of $D(A^*)$ for a given A generally is a non-trivial problem of great importance.

Definition 7. *A is called self-adjoint if $A = A^*$*

That means, in particular, that $D(A) = D(A^*)$ which is, generally, hard to prove.

Self-adjoint $\Rightarrow$ symmetric

Symmetric $\not\Rightarrow$ Self-adjoint If A is symmetric then $D(A) \subset D(A^*)$. But, generally, $D(A) \neq D(A^*)$

BUT

For bounded operators these notions coincide

Excercise 5 Find the historical anecdote about a funny dialogue between Friedrichs (a prominent mathematician) and Heisenberg (a genial physicist, the founder of quantum mechanics). See Peter Lax book on Functional Analysis[1].

Proposition 4. *1. A^* is closed*

[1] Solution: "The theory of self-adjoint operators was created by von Neumann to fashion a framework for quantum mechanics. The operators in Schrödinger's theory that are associated with atoms are partial differential operators whose coefficients are singular at certain points; these singularities correspond to the unbounded growth of the force between two electrons that approach each other. To define such differential operators as self-adjoint ones is not a trivial task $< \cdots >$. I recall in the summer of 1951 the excitement and elation of von Neumann when he learned that Kato has proved the self-adjointness of the Schrödinger operator associated with the helium atom.

And what do the physicists think of these matters? In the 1960s Friedrichs met Heisenberg, and used the occasion to express to him the deep gratitude of the community of mathematicians for having created quantum mechanics, which gave birth to the beautiful theory of operators in Hilbert space. Heisenberg allowed that this was so; Friedrichs then added that the mathematicians have, in some measure, returned the favor. Heisenberg looked noncommittal, so Friedrichs pointed out that it was a mathematician, von Neumann, who clarified the difference between a self-adjoint operator and one that is merely symmetric. "What's the difference," said Heisenberg."

2. $A_1 \subset A_2 \implies A_2^* \subset A_1^*$

3. *if A is closable then $(\bar{A})^* = A^*$*

   Easier items:

4. $(\lambda A)^* = \bar{\lambda} A^*$

5. $(A + \textit{bounded})^* = A^* + (\textit{bounded})^*$

Proof. 1): Simple play with definitions.
Let $x_n \in D(A^*)$ and $x_n \to x_0$; $A^* x_n \to y_0$.
That means that:
$\forall x \in D(A)$ one has

$$(Ax, x_n) = (x, A^* x_n)$$

Passing to the limit $n \to \infty$ gives

$$(Ax, x_0) = (x, y_0)$$

Thus, $x_0 \in D(A^*)$ and $A^* x_0 = y_0$ and A^* is closed.

2): Again simple play with definitions.
Let $z \in D(A_2^*)$.
Then $\forall x \in D(A_2)$

$$(A_2 x, z) = (x, A_2^* z)$$

But $A_2 x = A_1 x$ for all $x \in D(A_1) \subset D(A_2)$.
So $\forall x \in D(A_1)$

$$(A_1 x, z) = (x, A_2^* z)$$

Thus, $z \in D(A_2^*)$ and $A_2^* z = A_1^* z$.
3): Clearly $A \subset \bar{A}$, so by 2) $(\bar{A})^* \subset A^*$.
It remains to prove that $A^* \subset (\bar{A})^*$.
Let $y \in D(A^*)$. Then $\forall x \in D(A)$

$$(Ax, y) = (x, A^* y). \tag{1.1}$$ $\boxed{\mathtt{A}}$

We have to prove that
$\forall z \in D(\bar{A})$

$$(\bar{A} z, y) = (z, (\bar{A})^* y)$$

or, what is the same,

$$(\bar{A} z, y) = (z, A^* y)$$

Let $x_n \in D(A)$; $x_n \to z$ $Ax_n \to \bar{A} z$. Take $x = x_n$ in (1.1) and pass to the limit.
Exercise 6: Prove 4) and 5)

1.4 von Neumann's Graph Technique

Informal remark

NOT A THEOREM: $\Gamma(A) \perp \Gamma(A^*)$; $\Gamma(A) \oplus \Gamma(A^*) = H \oplus H$. **BUT** that is what one has to keep in mind as "almost true". Can be improved and upgraded to a correct statement.

Reminder: Let L an arbitrary linear subspace of H. $L^\perp$ is closed; $(L^\perp)^\perp = \bar{L}$; $H = \bar{L} \oplus L^\perp$.

$y \in D(A^*)$ iff $\forall x \in D(A)$

$$(Ax, y) = (x, A^*y)$$

or

$$(Ax, y) - (x, A^*y) = 0$$

or

$$\langle [Ax, -x], [y, A^*y] \rangle_{H \oplus H} = 0$$

or

$$\langle U[x, Ax], [y, A^*y] \rangle_{H \oplus H} = 0\,,$$

where

$$U : H \oplus H \to H \oplus H$$

is the unitary operator (introduced by von Neumann) defined via

$$U([u, v]) = [v, -u]$$

Remark. Unitarity implies

$$U(L^\perp) = (U(L))^\perp$$

and

$$U(\bar{S}) = \overline{U(S)}$$

(here L is a linear subspace, S is an arbitrary subset of $H \oplus H$.)

Remark. Clearly $U^2 = -I$ and, therefore, $U^2(L) = L$ for any linear subspace of $H \oplus H$.

As we showed: $y \in D(A^*)$ iff $\forall x \in D(A)$

$$\langle U[x, Ax], [y, A^*y] \rangle_{H \oplus H} = 0\,.$$

Thus,

$$\boxed{\Gamma(A^*) = [U(\Gamma(A)]^\perp = U(\Gamma(A)^\perp) \qquad (1.2)}$$ B

and

$$H \oplus H =$$
$$\overline{U(\Gamma(A))} \oplus [U(\Gamma(A))]^{\perp} =$$
$$= U(\overline{\Gamma(A)}) \oplus \Gamma(A^*) =$$

(apply U to both sides)

$$= \overline{\Gamma(A)} \oplus U(\Gamma(A^*))$$

Again:

$$\boxed{H \oplus H = \overline{\Gamma(A)} \oplus U(\Gamma(A^*))} \tag{1.3}$$

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and that is true even for nonclosable A!

Proposition 5. *A^* is densely defined $\Longleftrightarrow$ A is closable.*

In this case

$$(A^*)^* = \bar{A}$$

Reminder: Adjoint A^* is defined only for densely defined A. But A^* need not be densely defined !

Proof.

END OF LECTURE 2

LECTURE 3

1. $\Longrightarrow$

We have to show that $\overline{\Gamma(A)}$ is a graph.

In fact,

$$\overline{\Gamma(A)} = ((\Gamma(A))^{\perp})^{\perp} = \left[U^2\Gamma(A)\right]^{\perp\perp} =$$
$$= \left[U\left([U(\Gamma(A))]^{\perp}\right)\right]^{\perp} =$$

using (1.2)

$$= [U\Gamma(A^*)]^{\perp} =$$

(since A^* is densely defined, A^{**} exists and we can use (1.2) once again)

$$= \Gamma(A^{**})$$

and that is a graph ! (of A^{**})

2. $\Longleftarrow$

Let $h \perp D(A^*)$ and $h \neq 0$. This gives a contradiction. In fact $(h, 0) \perp \Gamma(A^*)$, therefore,

$$(0, -h) \perp U\Gamma(A^*)$$

and, therefore,

$$(0, h) \in (U(\Gamma(A^*))^{\perp} = \overline{\Gamma(A)}$$

due to (1.3). This means that $\overline{\Gamma(A)}$ is not a graph and A is not closable, which gives the needed contradiction.

1.5 Self-adjoint from closed: important general construction

Proposition 6. *Let A is densely defined and closed. Then A^*A is self-adjoint.*

Proof. Since $\Gamma(A)$ is closed we have

$$\Gamma(A) \oplus U(\Gamma(A^*)) = H \oplus H$$

Thus, $\forall u, v \in H \; \exists x \in D(A), y \in D(A^*)$:

$$[u, v] = [x, Ax] + [A^*y, -y]$$

or, what is the same, the system

$$\begin{cases} u = x + A^*y \\ v = Ax - y \end{cases}$$

is solvable for any u and v in H. In particular, for $v = 0$. We have:

$$\begin{cases} u = x + A^*y \\ Ax = y \end{cases}$$

is solvable for any $u \in H$ (i. e. has solutions $x \in D(A); y \in D(A^*)$.

Notation: $\text{Image}\,(A) = \text{Range}\,(A) =: R(A)$

So

$$\boxed{R(I + A^*A) = H}$$

Lemma 1. *One has*

- *$D(I + A^*A)$ is dense*
- *$I + A^*A$ is symmetric*

Reminder: $D(I + A^*A) = D(A^*A) \cap D(I) = D(A^*A) = \{x \in D(A) : Ax \in D(A^*)\}$

Proof:

Density: Let $h \perp D(I + A^*A)$. Then $\exists x_0 \in D(I + A^*A) : h = (I + A^*A)x_0$. Therefore, since $Ax_0 \in D(A^*)$,

$$0 = ((I + A^*A)x_0, x_0) = (x_0, x_0) + (Ax_0, Ax_0) \geq 0$$

And $x_0 = 0$. Thus $h = 0$.

Symmetry: let $x \in D(I + A^*A)$. Then

$$((I + A^*A)x, x) = (x, x) + (Ax, Ax) \in \mathbb{R}$$

Thus, by the polarization identity (more precise, by the Exercise after polarization identity) $I + A^*A$ is symmetric.

Now Proposition immediately follows from the following criterion of self-adjointness (take $B = I + A^*A$ and $\lambda = 0$).

A useful criterion of self-adjointness

$$\boxed{\begin{array}{c} B - \text{densely defined, symmetric} \\ \\ (\exists\lambda \in \mathbb{C} : R(B - \lambda I) = R(B - \bar{\lambda} I) = H) \implies B \ \ is \ \ self - adjoint \end{array}}$$

Proof. One has to prove that $D(B^*) \subset D(B)$ (reminder: $D(B) \subset D(B^*)$ for symmetric B).

Let $x_0 \in D(B^*)$.

Then one can solve for $x \in D(B)$ the equation

$$(B - \lambda I)x = (B^* - \lambda I)x_0$$

Then for any $y \in D(B)$:

$$((B - \lambda I)x, y) = ((B^* - \lambda I)x_0, y) =$$

(since $x_0 \in D(B^*)$)

$$= (x_0, (B - \bar{\lambda} I)y)$$

But due to symmetry of B

$$((B - \lambda I)x, y) = (x, (B - \bar{\lambda} I)y)$$

Therefore, $\forall y \in D(A)$

$$(x, (B - \bar{\lambda} I)y) = (x_0, (B - \bar{\lambda} I)y)\,.$$

Since $R(B - \bar{\lambda} I) = H$, one has $x = x_0$ and, therefore, $x_0 \in D(A)$.

Important exercises

1. Let A be densely defined. Prove that

$$\boxed{\begin{array}{c} \text{1) Ker } A^* \text{ is closed} \\ 2) H = \text{Ker } A^* \oplus \overline{R(A)} \end{array}}$$

2. Let A be closed and densely defined.

   Assume that $\exists A^{-1}$ (generally, unbounded) and $\overline{R(A)} = \overline{D(A^{-1})} = H$. Then

$$\boxed{(A^*)^{-1} = (A^{-1})^*}$$

3. let A be closed and symmetric. Then

$$\boxed{A^* \text{ is symmetric } \Longrightarrow A \text{ is self-adjoint.}}$$

4. Prove that if A is symmetric but not self-adjoint then $\mathbb{R} \subset \mathrm{Spec}\,(A)$.
5. If A symmetric, $\overline{D(A)} = H$ then A is closable (already proved, give another proof using A^*).
6. I f A symmetric, $\overline{D(A)} = H$ then $\bar{A}$ is symmetric.

Solutions

1.1)
$y_n \to y$, $y_n \in \mathrm{Ker}\, A^*$; $\forall x \in D(A)$

$$(Ax, y_n) = (x, A^* y_n) = 0$$

and, therefore, $(Ax, y) = 0$. Thus, $A^* y = 0$.

1.2)
a)Orthogonality: $R(A) \perp \mathrm{KerT}^*$. Let $y \in \mathrm{KerT}^*$ then

$$(Ax, y) = (x, A^* y) = (x, 0) = 0\,.$$

b) One has

$$H = \overline{R(A)} \oplus (\overline{R(A)})^{\perp} =: \overline{R(A)} \oplus L$$

.

(!): $\mathrm{Ker}\, A^* \subset L$ (already done = a)); $L \subset \mathrm{Ker}\, A^*$:
Let $u \in L$ then $\forall x \in D(A)\ \ 0 = (Ax, u) = (x, 0)$ i. e. $A^* u = 0$.

2) Since $R(A) = D(A^{-1})$ is dense, $(A^{-1})^*$ exists. From Problem 1):

$$H = \overline{R(A)} \oplus \mathrm{Ker}\, A^* = H \oplus \mathrm{Ker}\, A^*$$

and $\mathrm{Ker}\, A^* = 0$, i. e. A^* has inverse.

Now $\forall x \in D(A), y \in D((A^{-1})^*$

$$(x, y) = (A^{-1}Ax, y) = (Ax, (A^{-1})^* y) =$$

(bounded w. r. t. x!!)

$$= (x, A^*(A^{-1})^* y)$$

Since $D(A)$ is dense, this implies $\forall y \in D((A^{-1})^*\ \ y = A^*(A^{-1})^* y$

Passing to $A := A^{-1}$, $A^{-1} := A$, one gets $\forall z \in D(A^*)\ \ z = (A^{-1})^* A^* z$.

3)$A \subset A^*$, therefore $A = \bar{A} = (A^*)^* \subset A^*$, and $A = A^*$.

4)Let some *real* λ do not belong to the spectrum. Then due to the above criterion of self-adjointness A is self-adjoint. Contradiction.

5)$\forall x, y \in D(A)\ \ (Ax, y) = (x, Ay)$ and, therefore, $y \in D(A^*)$ and $D(A) \subset D(A^*)$. Thus, $D(A^*)$ is dense and, therefore, A is closable.

6)Let $x_0, y_o \in D(\bar{A})$. Then $x_n \to x_0$, $Ax_n \to \bar{A}x_0$; $y_n \to y_0$, $Ay_n \to \bar{A}y_0$ for some sequences x_n, y_n from $D(A)$. Passing to the limit in

$$(Ax_n, y_n) = (x_n, Ay_n)$$

one gets

$$(\bar{A}x_0, y_0) = (x_0, \bar{A}y_0)$$

2 First Examples

2.1 Basic example: $-i\frac{d}{dx}$

Reminder:

1) $f : [a, b] \to \mathbb{R}$ is called absolutely continuous (a. c.) if

$$\forall \epsilon > 0 \exists \delta :$$

$$\forall a_1, b_1, \dots a_n, b_n \in [a, b] : a \le a_1 < b_1 \le \dots \le a_n < b_n \le b$$

$$\sum_{k=1}^{n} (b_k - a_k) < \delta \implies \sum_{k=1}^{n} |f((b_k) - f(a_k)| < \epsilon$$

2) f is a. c. on $[a, b]$ iff

$$f(x) = \int_a^x h(t)dt$$

for some $h \in L^1[a, b]$.

3) If $f(x)$ is a. c. then $f'(x)$ exists a. e. and $f'(x) = h(x)$ a. e.

4) If $f(x)$ is a. c. and $f'(x) = 0$ a. e. then $f(x) = C$ a. e.

Consider the operator

$A = -i\frac{d}{dx} : L_2(\mathbb{I}) \to L_2(\mathbb{I})$ with domain

$$D(A) = \{f \ \text{ a. c. on } \ \mathbb{I} : f' \in L_2(\mathbb{I}), f(0) = f(1) = 0\} =: H_0^1(\mathbb{I})$$

Remark: "-" from tradition: because $F^{-1}_{x \to y} x F_{z \to x} f(z) = -i\frac{d}{dy} f(y)$ in $L_2(\mathbb{R})$.

Integration by parts shows that A is symmetric. Clearly, $D(A)$ is dense in $L_2(\mathbb{I})$.

Reminder: $L_2(\mathbb{I}) \subset L_1(\mathbb{I})$:

$$\int_0^1 |u| \le \left(\int_0^1 |u|^2\right)^{1/2} \left(\int_0^1 1\right)^{1/2}$$

Introduce also the space

$$H^1(\mathbb{I}) = \{f \ \text{ a. c. on } \ \mathbb{I} : f' \in L_2(\mathbb{I})\}$$

Clearly, $\forall f, g \in H^1(\mathbb{I})$

$$(f', g) + (f, g') = f\bar{g}\Big|_0^1$$

and $-i\frac{d}{dx}$ is NOT symmetric on $H^1(\mathbb{I})$.

Proposition 7. *One has*

1. $D(A^*) = H^1(\mathbb{I})$
2. $(A^*)^* = A$

(i. e. A is closed).

Proof.

Lemma 2.

$$(R(A))^{\perp} = \mathbb{C}$$

Proof of the lemma. $\supset$: For $f \in H_0^1$

$$\int_0^1 Af\bar{1} = -i\int_0^1 f' = -if|_0^1 = 0$$

$\subset$:

$$A \subset B \implies B^{\perp} \subset A^{\perp}$$

So,

$$\mathbb{C}^{\perp} \subset R(A) \implies R(A)^{\perp} \subset \mathbb{C}^{\perp\perp} = \mathbb{C}$$

and it is enough to prove that $\mathbb{C}^{\perp} \subset R(A)$.

Let $f \in L_2$ $f \perp 1$. Then $f(x) = \frac{d}{dx}\int_0^x f$ $(L_2 \subset L_1)$ and $F(x) = \int_0^x f \in H_0^1$ due to $\int_0^1 f = 0$ and lemma is proved.

End of Lecture 3

Lecture 4

1) $D(A^*) = H^1$.

$$H^1 \subset D(A^*):$$

Let $g \in H^1$, $f \in D(A)$

$$(Af, g) = \int_0^1 (-i)f'\bar{g} = \int_0^1 f\overline{(-ig')}$$

Thus, $g \in D(A^*)$ and $A^*g = -ig'$.

$$D(A^*) \subset H^1:$$

Let $g \in D(A^*)$, $f \in D(A)$

$$(Af, g) = (f, A^*g) = (f, \frac{d}{dx}\int_0^x A^*g) =$$

by parts

$$= -(f', \int_0^x A^*g) = (-if', i\int_0^x A^*g) = (Af, i\int_0^x A^*g)$$

Thus, $\forall f \in D(A)$

$$(Af, g - i\int_0^x A^*g) = 0$$

and, $g - i\int_0 xA^*g \perp R(A)$ using the lemma, one gets

$$g - i\int_0 xA^*g = C$$

and

$$g = C + i\int_0^x A^*g \in H^1$$

($A^*g \in L_2 \subset L_1$!).

2)$(A^*)^* = A$.

1)$A \subset (A^*)^* (= \bar{A})$.

2)$(A^*)^* \subset A$:

One has

$$A \subset A^* \implies (A^*)^* \subset A^*$$

Thus $D((A^*)^*) \subset H^1$ and $(A^*)^*g = -ig'$.

Let $f \in D(A^*)$, $g \in D((A^*)^*)$. Then

$$(A^*f, g) = (f, (A^*)^*g) = (f, -ig') = (-if', g) + i(f\bar{g})\Big|_0^1$$

On the other hand

$$(A^*f, g) = (-if', g)$$

and, therefore

$$f(1)\bar{g}(1) - f(0)\bar{g}(0) = 0$$

(for any f in H^1)

Thus, $g(0) = g(1) = 0$ and $g \in H_0^1 = D(A)$.

Remark. One can start from $D(A) := C_0^\infty(\mathbb{I})$ and then take a closure. This motivates the following definition.

Definition 8. *Let $D < D(A)$. D is called a core of A if the closure of D in the graph norm $|||\cdot|||$ coincides with $D(A)$.*

In other words $\forall [x, Ax] \in \Gamma(A) \;\; \exists x_n \in D:$

$$[x_n, Ax_n] \to [x, Ax]\,.$$

Clearly, $\overline{A|D} = \bar{A}$.

Proposition 8.

$$C_0^\infty(\mathbb{I}) := \{f \in C^\infty(\mathbb{I}), f(0) = f(1) = 0\}$$

is a core of A.

Proof. Let $g \in D(A)$. (In particular, $g(0) = g(1) = 0$.) Then $g' \in L_2$ and $\exists f_n \in C_0^\infty(\mathbb{I}):\;\; f_n \to g'$ in L_2.

The problem is that $\int_0^x f_n$ is not in H_0^1 (although C^∞). Consider

$$G_n(x) = \int_0^x f_n - x\int_0^1 f_n \in H_0^1 = D(A)$$

Then

$$G_n' = f_n - \int_0^1 f_n \to g' - \int_0^1 g' = g' - g(1) - g(0) = g'$$

(convergence in L_2)

$$G_n \to \int_0^x g' + x(g(1) - g(0)) = g$$

(again in L_2).

Summarizing: $A = \bar{A} \subset A^*$; $D(A) = H_0^1$; $D(A^*) = H^1$; $D(A) \neq D(A^*)$.

We will find *a family* of operators B such that:

$$A \subset B = B^* \subset A^*$$

These are called *self-adjoint extensions of* A.

Exercise

For $z \in \mathbb{C} \setminus \{0\}$ let A_z be the operator $-i\frac{d}{dx}$ in $L_2(\mathbb{I})$ with domain

$$D(A_z) = \{f \in H^1 : f(1) = zf(0)\}.$$

Prove that

$$(A_z)^* = A_{1/\bar{z}}$$

Solution

Clearly, $\forall f \in D(A_z)$, $\forall g \in H^1$

$$(Af, g) = (f, -ig') + \frac{1}{i}f(0)\left(z\overline{g(1)} - \overline{g(0)}\right). \qquad (2.1) \quad \boxed{\texttt{parts}}$$

1)$D(A_z^*) \subset D(A_{1/\bar{z}})$.

Let $g \in D(A_z^*)$. Clearly, $A \subset A_z$ and, therefore, $A_z^* \subset A^*$. So, for $g \in D(A_z^*)$ the adjoint operator acts in the standard way: $A_z^* g = -ig'$.

Thus, for any $f \in D(A_z)$

$$0 = (A_z f, g) - (f, A^* g) = \frac{1}{i}f(0)[z\overline{g(1)} - \overline{g(0)}]$$

In particular one can take $f(x) = e^{\lambda x}$ with $e^\lambda = z$. Since $f(0) \neq 0$,

$$g(1) = \frac{1}{\bar{z}}g(0)$$

and $g \in D(A_{1/\bar{z}})$.

2)$D(A_{1/\bar z}) \subset D(A_z^*)$.

Let $g \in D(A_{1/\bar z})$. Then the second term in the r. h. s. of (2.1) is zero. Thus, $\forall f \in D(A_z)$

$$(A_z f, g) = (f, -ig')$$

and, therefore, $g \in D(A_z^*)$.

Corollary.

$$\boxed{|z| = 1 \implies A_z^* = A_z}$$

Thus,

$$A \subset A_{e^{i\theta}} = A_{e^{i\theta}}^* \subset A^* .$$

We will prove later that

1)All self-adjoint extensions of A have this form.

2)For different $\theta \in [0, 2\pi)$ the spectra of $A_{e^{i\theta}}$ are different.

Definition 9. *A is essentially self-adjoint if $\bar A$ is self-adjoint.*

Remark. In this case $\bar A$ is the unique s. a. extension of A. (**Exercise:** Explain !)

Information

$$-i\frac{d}{dx}$$

1)In $L_2(0, +\infty)$ with $D(A) = C_0^\infty(0, +\infty)$ is symmetric but has no self-adjoint extensions.

2)In $L_2(-\infty, \infty)$ with $D(A) = C_0^\infty(\mathbb{R})$ is essentially self-adjoint. Spectrum of the closure is the whole real axis $\mathbb{R}$.

2.2 Operators with complete system of eigenfunctions

Proposition 9. *Let A be symmetric, $\overline{D(A)} = H$, $\exists \{f_n\}_{n=1}^\infty \quad : \quad Af_n = \lambda_n f_n$; $f_n \in D(A)$ and $\{f_n\}$ is an orthonormal basis of H.*

Then A is essentially self-adjoint.

Proof.

Introduce the operator

$$\hat A : H \supset D(\hat A) \to H$$

with domain

$$D(\hat A) := \{\sum \alpha_k f_k : \sum |\alpha_k|^2 < \infty; \ \ \sum |\lambda_k \alpha_k|^2 < \infty\}$$

acting via

$$\hat A(\sum \alpha_k f_k) = \sum \lambda_k \alpha_k f_k$$

We are to prove that

1. $A \subset \hat{A}$
2. $\hat{A}$ is closed
3. $\bar{A} = \hat{A}$
4. $(\hat{A})^* = \hat{A}$

1. $A \subset \hat{A}$:

Let $u = \sum \alpha_k f_k \in D(A)$. Then (Bessel) $\sum |\alpha_k|^2 = ||u||^2 < \infty$. Moreover,

$$(Au, f_k) = (A(\sum \alpha_j f_j), f_k) =$$

(symmetry)

$$= (\sum \alpha_j f_j, Af_k) = (\sum \alpha_j f_j, \lambda_k f_k) = \lambda_k \alpha_k$$

(Clearly, $\lambda_k \in \mathbb{R}$). Thus, $||Au||^2 = \sum |\lambda_k \alpha_k|^2 < \infty$.

2. $\hat{A}$ is closed.

Reminder: $\underline{Spec(\hat{A}) \neq \mathbb{C}} \implies \hat{A}$ is closed.

Let $\Lambda = \overline{\cup_{k=1}^{\infty}\{\lambda_k\}}$. Since all λ_k are real, Λ^c is not empty. We will show that

$$z \in \Lambda^c \implies z \notin \text{Spec}\, \hat{A}$$

(i. e. $\exists (\hat{A} - zI)^{-1}$ bounded and defined everywhere)

Introduce $B : H \to H$:

$$Bu = B(\sum \alpha_n f_n) := \sum \frac{\alpha_n}{\lambda_n - z} f_n$$

where u is an arbitrary element of H; $||u||^2 = \sum |\alpha_n|^2 < \infty$. Since $\sup_k |\lambda_k - z| \geq \epsilon > 0$, the L_2-sum at the right is well-defined:

$$\sum \left| \frac{\alpha_n}{\lambda_n - z} \right|^2 < \infty\,.$$

Clearly, B is bounded ($||B|| \leq \epsilon^{-1}$).

Moreover, $R(B) = D(\hat{A})$:

$\subset$ - obvious (check the estimate!)

$\supset$:

Let $v = \sum \alpha_n f_n$ with $\sum |\alpha_n|^2 < \infty$ and $\sum |\lambda_n \alpha_n|^2 < \infty$ (i. e. $v \in D(\hat{A})$).

Clearly,

$$v = B(\sum \alpha_n (\lambda_n - z) f_n)$$

and the r. h. s. is well-defined since

$$\sum |\alpha_n(\lambda_n - z)|^2 \leq 2 \sum (|\alpha_n \lambda_n|^2 + |z|^2 |\alpha_n|^2) < \infty\,.$$

Now

$$B(\hat{A} - zI) = (\hat{A} - zI) = I$$

and $z \notin \operatorname{Spec} \hat{A}$.

3. $\hat{A} = \bar{A}$

This is the simplest part:

Let $P = [\sum \alpha_k f_k, \sum \lambda_k \alpha_k f_k] \in \Gamma(\hat{A})$ Then $P = \lim P_n$ in $H \oplus H$, where $P_n \in \Gamma(A)$ with

$$P_n = [\sum_1^n \alpha_k f_k, \sum_1^n \lambda_k \alpha_k f_k]$$

4. $(\hat{A})^* = \hat{A}$

It suffices to show that $(\hat{A})^* \subset \hat{A}$.

Let $g \in D(\hat{A})^*$.

Then

$$(\hat{A} f_n, g) = (f_n, (\hat{A})^* g) =: c_n$$

Since c_n is the n-th Fourier coefficient of $(\hat{A})^* g \in H$ one has

$$\sum |c_n|^2 \leq \infty$$

On the other hand

$$c_n = (\hat{A} f_n, g) = \lambda_n \alpha_n \,,$$

where α_n is the n-th Fourier coefficient of $g \in H$. Thus $g = \sum \alpha_n f_n$ with $\sum |\alpha_n|^2 < \infty$ and $\sum |\lambda_n \alpha_n|^2 < \infty$. Therefore, $g \in D(\hat{A})$.

EXAMPLES

2.2.1 Dirichlet problem on $\mathbb{I}$

$A = -\frac{d^2}{(dx)^2}$ in $L_2(\mathbb{I})$; $D(A) = \{f \in C^2(\mathbb{I}) : f(0) = f(1) = 0\}$

$$f_n(x) = \sin n\pi x$$

A is essentially self-adjoint.

2.2.2 Harmonic oscillator

$A = -\frac{d^2}{(dx)^2} + x^2$ in $L_2(\mathbb{R})$.; $D(A) = S(\mathbb{R})$ (rapidly decreasing C^∞-functions).

$$f_k(x) = H_k(x) e^{-\frac{x^2}{2}}$$

where $H_k(x) = (-1)^k e^{x^2} \left(\frac{d}{dx}\right)^k e^{-x^2}$ is the k-th Hermite polynomial.

REMARK

Everything could be checked **by bare hands**.

1)f_k are eigenfunctions: direct simple calculation.

2)Orthogonality: follows from symmetry of the operator

3)Completeness is the only difficulty. Let f is orthogonal to all f_k.

Obviously the linear span of the first k Hermite polynomials is the space of all polynomials of degree n. Thus,

$$\int_{-\infty}^{+\infty} f(x)x^n e^{-x^2/2}dx = 0$$

for all $n = 0, 1, \dots$ Consider

$$F(z) = \int_{-\infty}^{+\infty} e^{zx} f(x)e^{-x^2/2}dx$$

Then $F(z)$ is entire and

$$F(z) = \sum_{n=0}^{\infty} \frac{1}{n!} \left(\int_{-\infty}^{+\infty} x^n f(x)e^{-x^2/2}dx \right) z^n = 0$$

In particular, $F(-it) = 0$ for any real t. But

$$F(-it) = \int_{-\infty}^{+\infty} e^{-itx} f(x)e^{-x^2/2}dx$$

is the Fourier transform of $f(x)e^{-x^2/2}$. Thus, $f = 0$.

A is essentially self-adjoint.

End of Lecture 4

Lecture 5

Remark:

RELATIVELY HARD THEOREM (H. Weyl):

$$A = -\frac{d^2}{(dx)^2} + q(x)$$

with $D(A) = C_0^\infty(\mathbb{R})$ is essentially self-adjoint in $L_2(R)$ if (continuous) q is bounded from below.

HARD THEOREM (D. B. Sears): If $q(x) \geq -Q(x)$ with positive, even, not decreasing (for $x \geq 0$) function Q such that

$$\int_{-\infty}^{+\infty} \frac{dx}{\sqrt{Q(x)}} = \infty$$

then A is essentially self-adjoint.

Proposed as theme for a presentation at the end of the term.

2.3 Main example of the course: multiplication operator

(M, Ω, μ) - measure space (Ω is a σ-algebra; μ is a positive measure).

$$a : M \to \overline{\mathbb{R}}$$

is a measurable function such that $|a(m)| < +\infty$ a. e. (almost everywhere).

$$A : L_2(M, \mu) \supset D(A) \to L_2(M, \mu)$$

$$D(A) = \{f \in L_2(M, \mu) : af \in L_2(M, \mu)\}$$

$$Af = af$$

Proposition 10. *The operator A is self-adjoint.*

Proof.

1. $D(A)$ is dense

Let $f \perp D(A)$. Let $\chi_{\{|a|<N\}}$ be the characteristic function of the set

$$\{m \in M : |a(m)| < N\}$$

Clearly,

$$\chi_{\{|a|<N\}} f \in D(A)$$

(because $a\chi_{\{|a|<N\}} f$ =is the product of a bounded function ($a\chi_{\{|a|<N\}}$) and a function from L_2 (f) and, therefore, belongs to L_2).

Thus, for any N

$$0 = \int_M f \overline{\chi_{\{|a|<N\}} f} = \int_{|a|<N} |f|^2$$

and, therefore, $f = 0$ a. e.

2. $A = A^*$ Let $g \in D(A^*)$. Then $\forall F \in D(A)$

$$(Af, g) = (f, A^* g)$$

or

$$\int_M af\bar{g} = \int f\overline{A^* g}$$

Take $f := \chi_{\{|a|<N\}} h$ with an arbitrary $h \in L_2$. (As explained above $f \in D(A)$.) This gives

$$\int_{|a|<N} a\bar{g}h = \int_{|a|<N} \overline{A^* g} h$$

for any $h \in L_2$ and, therefore

$$a\bar{g} = \overline{A^* g}$$

on $|a| < N$ and, therefore, $a\bar{g} = \overline{A^* g}$ a. e. and, therefore, $ag \in L_2$ and $g \in D(A)$.

Thus $D(A^*) \subset D(A)$ and therefore, $A = A^*$.

INFORMATION

Very soon we will start proving THE MAIN THEOREM of the course (and one of the main theorems of the whole functional analysis): "the spectral theorem for self-adjoint operators": *any s. a. operator is unitary equivalent to the multiplication operator corresponding to some function $a(\cdot)$ and (M, Ω, μ).*

2.4 Friedrichs extension of a positive symmetric operator

Let A be symmetric, $\overline{D(A)} = H$, and let $\exists \gamma > 0$: $\forall u \in D(A)$

$$(Au, u) \geq \gamma^2 ||u||^2 \qquad \boxed{\texttt{bound}} \tag{2.2}$$

Energetic norm on $D(A)$:
Define

$$|||u|||^2 := (Au, u)$$

on $D(A)$.

Check axioms of norm:
1)$|||u||| = 0 \Longleftrightarrow u = 0$
2)$|||\lambda u||| = |\lambda| |||u|||$
3)$|||u + v||| \leq |||u||| + |||v|||$
Only the third is not immediately obvious. Fairly standard:

$$(A(u + tv), u + tv) = (Au, u)t^2 + 2\Re(Au, v)t + (Av, v) \geq 0\,,$$

therefore,

$$(\Re u, v)^2 \leq (Au, u)(Av, v) \qquad \boxed{\texttt{neq}} \tag{2.3}$$

and

$$|||u + v|||^2 = (A(u + v), u + v) = |||u|||^2| + |||v|||^2 + 2\Re(Au, v) \leq$$
$$|||u|||^2 + |||v|||^2 + 2|||u|||\ |||v||| = (|||u||| + |||v|||)^2$$

Remark. For future use let us notice that considering $(A(u + itv), u + itv)$, one shows that $(\Im(Au, v))^2 \leq (Au, u)(Av, v)$ which (together with (2.3)) gives

$$|(Au, v)|^2 \leq 2(Au, u)(Av, v)\,. \qquad \boxed{\texttt{neq1}} \tag{2.4}$$

So, one can consider completion, E, of $D(A)$ in $||| \cdot |||$. The main observation is the following: each element of this completion E can be identified with an element of the space H. More precisely, there is a continuous injection

$$j : E \to H$$

and E can be identified with a linear subspace of H dense in the standard $|| \cdot ||$ norm. For any $e \in E \equiv j(E) \subset H$ one has

$$||e||_H \leq \frac{1}{\gamma} |||e|||_E\,. \qquad \boxed{\texttt{in1}} \tag{2.5}$$

Now let us give the full details.
Due to (2.2) there is a map

$$\{\text{Cauchy sequence} \in D(A) \text{ w. r. t. } ||| \cdot |||\ \} \mapsto h \in H$$

One has to prove

1. Equivalent Cauchy sequences are mapped to the same $h \in H$
2. If two Cauchy sequences are mapped to the same $h \in H$ then they are equivalent

Proof:

1. This is trivial:

Let u_n, v_n two Cauchy sequences from $D(A)$ w. r. t. $|||\cdot|||$. Such that $u_n \sim v_n$ (i. e. $|||u_n - v_n||| \to 0$). Let $u_n \to h_1$, $v_n \to h_2$ in H. One has

$$||h_1 - h_2|| = ||h_1 - u_n + u_n - v_n - (h_2 - v_n)|| \leq ||h_1 - u_n|| + ||h_2 - v_n|| + ||u_n - v_n|| \leq$$

$$||h_1 - u_n|| + ||h_2 - v_n|| + \frac{1}{\gamma}|||u_n - v_n||| \to 0$$

as $n \to \infty$.

2. That is somewhat unexpected.

We have to prove that for a Cauchy sequence $w_n = u_n - v_n$ (w. r. t. $|||\cdot|||$)

$$||w_n|| \to 0 \implies |||w_n||| \to 0$$

but inequality (2.5) has the opposite direction! However:

$$\Big|\; |||w_n||| - |||w_m|||\; \Big| \leq |||w_n - w_m||| \to 0$$

and, therefore,

$$|||w_n||| \to \alpha$$

Now consider

$$(Aw_n, w_m)$$

One has

$$(Aw_n, w_m) = (Aw_n, w_n) + (Aw_n, w_m - w_n) = |||w_n|||^2 + (Aw_n, w_m - w_n) =$$

$$= \alpha + o(1) + (Aw_n, w_m - w_n)$$

Due to (2.4)

$$|(Aw_n, w_m - w_n)| \leq \sqrt{2}|||w_n|||\,|||w_m - w_n||| = \sqrt{2}(\alpha + o(1))o(1) = o(1)$$

as $n, m \to \infty$

On the other hand for a given n

$$(Aw_n, w_m) \to 0$$

as $n \to \infty$ (since $w_m \to 0$ in H). This implies $\alpha = 0$.

Example to keep in mind when thinking on energetic space

Let Ω be an open domain in $\mathbb{R}^n$; $A = -\Delta + 1$; $H = L_2(\Omega)$, $D(A) = C_0^\infty(\Omega)$.
For $u \in D(A)$ one has

$$(Au, u) = (-\Delta u, u) + (u, u) =$$

(since $u|\partial\Omega = 0$)

$$= (\nabla u, \nabla u) + (u, u)$$

Thus, the energetic space is the closure of $C_0^\infty(\Omega)$ in the norm

$$|||u|||^{1/2} = (\nabla u, \nabla u) + (u, u)$$

and that is the Sobolev space $H_0^1(\Omega)$.

Thus, we have a Hilbert space E

$$E \subset H$$

with norm

$$|||e||| = \lim_{n\to\infty} |||u_n|||$$

and hermitian product

$$\langle\!\langle e, f\rangle\!\rangle_E = \lim_{n\to\infty} (Au_n, v_n)$$

where u_n, v_n are $|||\cdot|||$-Cauchy sequences from $D(A)$ defining $e, f \in E$.

Exercise: Using polarization identity, prove independence of the hermitian product (and, therefore, the norm) from the choice of Cauchy sequences.

End of Lecture 5

Lecture 6

Friedrichs construction of a self-adjoint operator

General setting for Fiedrichs construction:

- two Hilbert spaces E and H with hermitian products $\langle\!\langle \cdot, \cdot\rangle\!\rangle_E$ and $(\cdot, \cdot)_H$
- continuous injection

$$j : E \to H$$

with dense range (so, one can identify E with a dense linear subspace of H).

Once one is in such set-up[2], Friedrichs construction leads to a linear self-adjoint operator in H.

We are to define the operator $\hat{A}$ (notation shows that it is related to the operator A that was used to construct E, but, in general, we are in the just described general setting (having no need in the initial operator A).

Step 1. Define

$$D(\hat{A}) := \{e \in E : \text{ the functional } \langle\!\langle \mathrm{e}, \cdot\rangle\!\rangle \text{ is bounded in H}\}$$

(i. e. $\langle\!\langle e, f\rangle\!\rangle \le C||f||_H$ for any $f \in E$, and , since E is dense in H the functional can be extended to all H as a bounded functional $l_e(\cdot)$).

Step 2.

[2]Equivalent point of view: Hilbert space H, positive closed sesquilinear form a ("closed" $\equiv$ "$D(a) = E$ is dense in H (w. r. t. norm in H) and complete w. r. t. the second hermitian product $\langle\!\langle, \cdot, \cdot\rangle\!\rangle = a(\cdot, \cdot)$)"; "positive" = "injection is continuous"="$a(u, u) \ge \gamma^2(u, u)$")

Riesz theorem implies

$$l_e(\cdot) = (z, \cdot)_H$$

for some $z \in H$. Define

$$\hat{A}(e) = z$$

Remark: For the special set-up arising from positive symmetric operator A

$$A \subset \hat{A}\,.$$

In fact, if $e \in D(A) \subset E$ then

$$|\langle\langle e, u\rangle\rangle| = |(Ae, u)| \leq C||u||_H$$

for any $u \in D(A)$ and, therefore, $e \in D(\hat{A})$ and $\hat{A}(e) = Ae$.

Properties of $\hat{A}$

Proposition 11. *One has*

1. $\hat{A} : D(\hat{A}) \to H$ *is a bijection. (In particular,* $D(\hat{A})$ *is much bigger than* $\{0\}$ *which is not immediately obvious in the general setting.)*
2. $(\hat{A})^{-1}$ *is bounded*

Proof.
1a) Injectivity:
Let $e \in D(\hat{A})$, so, $\langle\langle e, \cdot\rangle\rangle$ is bounded in H. Thus,

$$|||e; E|||^2 = \langle\langle e, e\rangle\rangle = (\hat{A}e, e) \leq ||\hat{A}e; H||||e; H|| \leq$$

($||je; H|| \leq C|||e; E|||$ for general setting, $C = \frac{1}{\gamma}$, $|||e||| \geq \gamma||e||$ for the special set-up related to a positive symmetric operator A)

$$\leq ||\hat{A}e; H||\,|||e; E|||\frac{1}{\gamma} \tag{2.6}$$ $\boxed{\texttt{cont}}$

and, therefore,

$$||\hat{A}e; H|| \geq \gamma|||e; E|||$$

which implies injectivity.

1b)**Surjectivity:**
Choose any $h \in H$. Then

$$(h, \cdot)_H$$

is an antilinear functional bounded not only in H but in E! Namely, for any $e \in E$

$$(h, e)_H \leq ||h; H||\,||e; H|| \leq ||h; H||\frac{1}{\gamma}|||e; E|||$$

(or, in the general set-up: for any $e \equiv je$

$$(h, je)_H \leq ||h; H|| \, ||je; H|| \leq ||h; H|| \, C|||e; E|||)$$

Thus, Riesz theorem (applied to E gives

$$(h, e) = \langle\!\langle \tilde{h}, e \rangle\!\rangle$$

for some $\tilde{h} \in E$ and any $e \in E$. Therefore $\tilde{h} \in D(\hat{A})$ and $\hat{A}(\tilde{h}) = h$.

2)Boundedness of $(\hat{A})^{-1}$ is equivalent to (2.6).

And now **the main statement:**

Proposition 12. *The operator $\hat{A}$ with domain $D(\hat{A})$ is self-adjoint:*

$$(\hat{A})^* = \hat{A}$$

Proof.

1) $\hat{A}$ is symmetric:

Let $e_1, e_2 \in D(\hat{A})$.
Then

$$\langle\!\langle e_1, e_2 \rangle\!\rangle_E = (\hat{A}e_1, e_2)_H = \overline{\langle\!\langle e_2, e_1 \rangle\!\rangle_E} = \overline{(\hat{A}e_2, e_1)_H} = (e_1, \hat{A}e_2)_H$$

2) $D((\hat{A})^*) \subset D(\hat{A})$:

Let $f \in D((\hat{A})^*$. Since $\hat{A} : D(\hat{A}) \to H$ is a surjection, one has

$$(\hat{A})^* f = \hat{A}e$$

for some $e \in D(\hat{A})$.

Now for any $w \in D(\hat{A})$

$$((\hat{A})^* f, w) = (f, \hat{A}w) = (\hat{A}e, w) =$$

due to symmetry of $\hat{A}$

$$= (e, \hat{A}w)$$

Thus,

$$(f, \hat{A}w) = (e, \hat{A}w)$$

for any $w \in D(\hat{A})$. Since $R(\hat{A}) = H$, we get $f = e$ and $f \in D(\hat{A})$.

Remark. Notice that we have just repeated the trick from the proof of criterion of self-adjointness from page 11. In fact, we could simply refer to this criterion with $B = \hat{A}$ and $\lambda = 0$.

Remark. Did you notice *that* the previous proof contained a serious gap?

Of course, if $\hat{A}$ came from A then, as we have showed, $D(\hat{A})$ contains $D(A)$ and, therefore, is dense.

But in general setting (when there is no operator A) part 2) of Proposition 12 makes no sense: we **did not prove** that $D(\hat{A})$ is dense and, therefore, had no right to introduce the operator $\hat{A}^*$!

So, let us accurately analyse definition of $D(\hat{A})$.

We have the bounded injection

$$j : E \to H$$

with dense range. Introduce the adjoint operator

$$j^* : H \to E$$

via

$$\langle\langle j^*h, e\rangle\rangle_E = (h, je)_H$$

(since, $||j^*h|| = \sup_{|||e|||=1}\langle\langle j^*h, e\rangle\rangle = \sup(h, je) \leq ||j||||h||$ it is bounded). Clearly, the range $R(j^*)$ is dense. (Assume that $e_1 \perp R(j^*)$, then for any $h \in H$

$$0 = (j^*h, e_1) = (h, je_1)$$

and $he_1 = 0$. Thus, $e_1 = 0$.)

This implies that the range $R(jj^*)$ of the composition $jj^* : H \to H$ is also dense:

(Let $h \in H$ then for any $\epsilon > 0$ $\exists e_0 \in E$ and $\delta > 0$ such that $||je_0 - h|| < \epsilon$ and for all $e \in E$ such that

$$|||e - e_0||| < \delta$$

one has

$$||j(e) - h|| < 2\epsilon$$

(density of $R(j)$ + continuity of j!)

But Rj^* is dense, so $\exists h_1 \in H$ such that $||j^*h_1 - e_0|| < \delta$. Thus,

$$||jj^*h_1 - h|| < 2\epsilon$$

and $R(jj^*)$ is dense.)

To prove that $D(\hat{A})$ is dense it remains to observe that

Lemma 3. *One has*

$$D(\hat{A}) = R(jj^*)$$

This is in fact a tautology (directly follows from the definitions).

$$D(\hat{A}) = \{h = je : \exists h_1 \in H \ \ \forall e_1 \in E \ \ \langle\langle e, e_1\rangle\rangle_E = (h_1, je_1)_H\}$$

Thus, $e = j^*h_1$ and $h = jj^*h_1$.

Self-adjoint operator of the Dirichlet problem

$$\begin{cases} -\Delta u + u = 0 & \text{in } \Omega \\ u\Big|_{\partial\Omega} = 0 \end{cases}$$

Let Ω be an open domain in $\mathbb{R}^n$; $A = -\Delta + 1$; $H = L_2(\Omega)$, $D(A) = C_0^\infty(\Omega)$.

$$E = H_0^1(\Omega)$$

$$\langle\langle u, v\rangle\rangle = (\nabla u, \nabla v) + (u, v)$$

$$D(\hat{A}) = \{u \in H_0^1(\Omega) : \exists f \in L_2(\Omega) : \forall w \in H_0^1(\Omega) \quad \langle\langle u, w\rangle\rangle = (f, w)\}$$

In other words:

$$D(\hat{A}) = \{u \in H_0^1(\Omega) : \Delta u \in L_2(\Omega)\}$$

(Elliptic theory: for smooth $\partial\Omega$: $u \in H_0^1(\Omega)$ and $\Delta u \in L_2$ (in the sense of $\mathcal{D}'(\Omega)$) then $u \in H^2(\Omega)$. In this case $D(\hat{A}) = H_0^1(\Omega) \cap H_2(\Omega)$)

Self-adjoint operator of the Neumann problem

$$\begin{cases} -\Delta u + u = 0 & \text{in } \Omega \\ \frac{\partial u}{\partial n}\Big|_{\partial\Omega} = 0 \end{cases}$$

Apply general Friedrichs construction to $(H^1(\Omega), L_2(\Omega))$ (no starting operator A). (Or, consider the closed positive form $a(u, v) = (\nabla u, \nabla v) + (u, v)$ with domain $H^1(\Omega)$.)

$$D(\hat{A}) = \{u \in H^1(\Omega) : \exists f \in L_2(\Omega) : \forall w \in H^1(\Omega) \quad \langle\langle u, w\rangle\rangle = (f, w)\}$$

It can be shown (using elliptic theory) that for smooth $\partial\Omega$:

$$D(\hat{A}) = \{u \in H^2(\Omega) : u_n|_{\partial\Omega} = 0\}$$

3 The spectral theorem

3.1 Resolvent

3.1.1 General properties of the resolvent

Let A be a closed (otherwise $\mathrm{Spec}\,(A) = \mathbb{C}$ and all the statements below are void (although correct)) operator, $A : H \supset D(A) \to H$.

Proposition 13. *The spectrum of A, $\mathrm{Spec}\,(A)$ is closed. If $z_0 \in \mathrm{Spec}\,(A)^c$ and $\alpha_0 = ||(A - z_0 I)^{-1}||$ then*

$$\mathrm{Spec}\,(A) \cap \{w : |w - z_0| < \frac{1}{\alpha}\} = \emptyset\,.$$

Proof. We start with formal calculation:

$$\frac{1}{A-z} = \frac{1}{A-z_0-(z-z_0)} = \frac{1}{(A-z_0)\left[I - \frac{z-z_0}{A-z_0}\right]} =$$

$$= \sum_{n=0}^{\infty} \left(\frac{1}{A-z_0}\right)^{n+1} (z-z_0)^n$$

Now let

$$B = (A - z_0 I)^{-1}$$

$$C := \sum_{n=0}^{\infty} B^{n+1}(z-z_0)^n$$

(clearly, the latter series converges for $|z-z_0| < \frac{1}{||B||}$)

Step 1.

Play with power series: $(w := z - z_0)$

$$wBC = wB^2 + w^2B^3 + \cdots = C - B = wCB$$

Remark. Compare this with resolvent identity from Proposition 14 below.

(In particular, one gets $\operatorname{Ker} C = \operatorname{Ker} B$ $(= \{0\}!)$; $R(C) = R(B)$ $(= D(A)!)$:
Since $wBC = C - B$, $\operatorname{Ker} C \subset \operatorname{Ker} B$ and $R(C) \subset R(B)$.
Since $wCB = C - B$, $\operatorname{Ker} C \supset \operatorname{Ker} B$ and $R(C) \supset R(B)$.)

Remark. Clearly, the above relations with Ker-s and R-s of the resolvent (see the definition below) are true for any z_0, z belonging to the same connected component of the complement of the spectrum.

Step 2.

We are to show that

$$C(A - zI) = (A - zI)C = I$$

Let $x \in D(A)$ and

$$(A - z_0 I)x = y$$

i. e.

$$x = By.$$

Let us use the relation

$$wCB = C - B\,.$$

with $w = z - z_0$. One has

$$wCx = wCBy = Cy - By = Cy - x$$

$$C(y - wx) = x$$

$$C((A - z_0)x - (z - z_0)x) = x$$

and

$$C(A - zI)x = x\,.$$

Prove $(A - zI)Cx = x$ as an exercise.

Resolvent (analytic operator-function on $(\operatorname{Spec} A)^c$):

$$R(z;A) := (A - zI)^{-1}$$

Resolvent Identity

Proposition 14.

$$R(z;A) - R(w;A) = (z-w)R(z;A)R(z;w) \qquad (3.1)$$ RI

and

$$R(z;A)R(z;w) = R(w;A)R(z;A)\,.$$

Proof.

$$z - w = (A - w) - (A - z)$$
$$(A-z)^{-1}(z-w)(A-w)^{-1} = (A-z)^{-1} - (A-w)^{-1}$$
$$(A-w)^{-1}(z-w)(A-z)^{-1} = (A-z)^{-1} - (A-w)^{-1}$$

End of lecture 6

Lecture 7

An example of application of the resolvent identity

Riesz projector. Let A be a closed operator and let $\gamma \subset \mathbb{C}$ be a positively oriented closed contour such that $\gamma \cap \operatorname{Spectrum}(A) = \emptyset$. Introduce the bounded operator $P : H \to H$ via

$$P := -\frac{1}{2\pi i} \oint R(z;A)dz$$

Then P is a projection, i. e. $P^2 = P$.

Proof. Let δ be a "slightly shrunken" γ still lying in the complement to the spectrum and not intersecting γ.

$$P^2 = \left(\frac{1}{2\pi i}\right)^2 \oint_\delta R(\zeta;A)\,d\zeta \oint_\gamma R(z;A)\,dz = \left(\frac{1}{2\pi i}\right)^2 \oint\oint_{\gamma\times\delta} R(\zeta;A)R(z;A)d\zeta dz =$$

(now use resolvent identity)

$$= \left(\frac{1}{2\pi i}\right)^2 \oint\oint_{\gamma\times\delta} \frac{1}{z-\zeta}(R(z;A) - R(\zeta;A))d\zeta dz =$$

$$= \left(\frac{1}{2\pi i}\right)^2 \left[\oint_\gamma R(z;A)dz \oint_\delta \frac{d\zeta}{z-\zeta} - \oint_\delta R(\zeta;A)d\zeta \oint_\gamma \frac{dz}{z-\zeta}\right] =$$

$$= \left(\frac{1}{2\pi i}\right)^2 \left[0 - \oint_\delta R(\zeta;A)d\zeta\,(2\pi i)\right] = P\,.$$

Proposition 15.

$$\frac{d}{dz}R(z;A) = \left[R(z;A)^2\right]^2$$

Proof.

$$R(z;A) = C = \sum_{n=0}^{\infty}(z-z_0)^n B^{n+1}$$

$$\frac{d}{dz}R(z;A) = \sum_{n=1}^{\infty} n(z-z_0)^{n-1}B^{n+1} =$$

$$\left(\sum_{n=0}^{\infty}(z-z_0)^n B^{n+1}\right)\left(\sum_{n=0}^{\infty}(z-z_0)^n B^{n+1}\right).$$

The last equality follows from the same equality for ordinary power series:

$$\frac{d}{d\zeta}\left(\frac{1}{b-\zeta}\right) = \left(\frac{1}{b-\zeta}\right)^2.$$

(Plug $\beta := b^{-1}$; since all the powers of the operator B commute, the identity with β remains true with β replaced by B.)

3.1.2 Resolvent of a self-adjoint operator

Proposition 16. *Let $A : H \supset D(A) \to H$ be a self-adjoint operator. Then*

1. $\operatorname{Spec} A \subset \mathbb{R}$
2. $R^*(z;A) = R(\bar{z};A)$ *for* $z \in \mathbb{C}\setminus\mathbb{R}$.
3. $$||R(z;A)|| \le \frac{1}{|\Im z|}$$

Proof.

Key calculation for self-adjoint operators:

Let $z = x+iy$; $|y| > 0$; $A = A^*$, $u \in D(A)$. Then (please, check!)

$$||(A-zI)u||^2 = ||(A-x)u||^2 + y^2||u||^2 \ge |y|^2||u||^2 \tag{3.2}$$

est1

Thus, $A - zI$ is an injection and $(A-zI)^{-1} : R(A-zI) \to H$ is bounded.

Lemma 4. *$R(A-zI)$ is dense, $\overline{R(A-zI)} = H$.*

In fact, let $g \perp R(A-zI)$ then for any $x \in D(A)$

$$0 = ((A-zI)x, g)$$

and $g \in \operatorname{Ker}(A-zI)^* = \operatorname{Ker}(A-\bar{z}I) = \{0\}$ due to (3.2).

$A = A^*$ is closed, therefore $\Gamma(A-zI)^{-1}$ is closed and, therefore, $R(A-zI)$ being dense should coincide with H (explain!). Thus $(A-zI)^{-1} : H \to H$ is bounded. So 1) and 3) is proved. 2) follows from exercise 2 on page 11.

3.2 Proof of the spectral theorem

3.2.1 Functions analytic on $\mathbb{C}_+$ with positive imaginary part

Let $A : H \supset D(A) \to H$, $A = A^*$, $f \in H$ and let

$$\Phi_f(z) := (R(z;A)f, f)\,.$$

Let $\mathbb{H}_+ = \{z \in \mathbb{C} : \Im z > 0\}$ (similarly for $\mathbb{H}_-$).

Lemma 5. *The function Φ_f satisfies*

1. Φ_f *is analytic in* $\mathbb{C} \setminus \mathbb{R}$
2. $\Phi_f : \mathbb{H}_\pm \to \mathbb{H}_\pm$ *(the most important!)*
3. $|\Phi_f(z)| \le \frac{||f||^2}{|\Im z|}$
4. $\overline{\Phi_f(z)} = \Phi_f(\bar z)$

Proof. 1) See Proposition 15.
2) Since $R(A - zI) = H$, $f = (A - zI)g$ for some $g \in H$

$$(R(z)f, f) = (g, (A - zI)g) = (g, Ag) - \bar z(g, g)$$

The first term at the right is real, the sign of the imaginary part of the second term is the same as the sign of $\Im z$.
3)Follows from (est1 3.2).
4)

$$\overline{\Phi_f(z)} = (f, R(z)f) = (R^*(z)f, f) = (R(\bar z)f, f) = \Phi_f(\bar z)\,.$$

Spectral Theorem

Theorem 1. *Let A be a self-adjoint operator in H. Then there exists a measure space (M, Ω, μ), a measurable real-valued function $f : M \to \mathbb{R}$ and a unitary operator $U : H \to L_2(M, \mu)$ such that $A = U^{-1} m_f U$, where m_f is the multiplication operator*

$$L_2(M, \mu) \ni u \mapsto m_f(u) \equiv fu \in L_2(M, \mu)\,.$$

Remark. How the measure μ appears? Informal answer: through

Herglotz theorem!

The analytic background: holomorphic function $\Phi_f(z)$ maps upper half-plane into itself: all such functions are well-understood analytically: Roughly speaking:

$$\phi_x : z \mapsto \frac{1}{x - z}$$

for $x \in \mathbb{R}$ is such a function; moreover, it is clear that any combination

$$\int_{\mathbb{R}} \phi_x d\sigma(x)$$

with finite Borel measure $d\sigma$ on $\mathbb{R}$ is such a function. Herglotz theorem says that any such function (if it vanishes at $+i\infty$) can be represented as the latter integral. The measure space Ω will be a countable disjoint union of real lines (each of them provided with its own Borel measure $d\sigma$ given by Herglotz theorem). The function $f(x)$ will coincide with $f(x) = x$ being restricted to each real line.

Remark. The most elegant proof (and the most natural) of the Herglotz theorem uses Krein-Milman theorem (roughly speaking, functions ϕ_x are extremal points of the set of all analytic functions in H with positive imaginary part). The most standard short modern proof is based on Banach–Alaoglu theorem (closed unit ball of the dual space of a normed vector space is compact in the weak$*$ topology). We will give an old-fashioned proof based on Helly theorem for Stieltjes integrals (which in fact is a specialization of Banach-Alaoglu).

Proposition 17. *(Herglotz Theorem.) Let $f : \mathbb{H}_+ \to \mathbb{H}_+$ is holomorphic and let*

$$|f(z)| \le \frac{1}{\Im z}\,.$$

Then there exists a monotone (non-decreasing) function[3] $\sigma : \mathbb{R} \to \mathbb{R}$ *such that*

$$0 \le \sigma(x) \le 1$$

$$\lim_{x\to-\infty} \sigma(x) = 0$$

$$\lim_{x\to+\infty} \sigma(x) \le 1$$

and

$$f(z) = \int_{\mathbb{R}} \frac{d\sigma(x)}{x - z}\,.$$

Proof of Herglotz Theorem

(was not shown in class)

Lemma 6. *(**Schwarz formula**) Let f be holomorphic in $\{|z| < R_1\}$ and $0 < R < R_1$. Then for any z, $|z| < R$ one has the representation*

$$f(z) = i\Im f(0) + \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{Re^{i\phi} + z}{Re^{i\phi} - z}\Re f(Re^{i\phi})\, d\phi\,. \qquad (3.3) \quad \boxed{\texttt{Sch}}$$

Proof. That is a well-known consequence of the classical Poisson formula (see, e. g., Ahlfors course in CA), but can be easily proved directly. Analytical reason:

$$\oint_{|z|=R} \bar{z}^k z^m\, dz = 0$$

for $m + 1 - k \neq 0$.

[3]continuous from the left, i. e. defining Borel measure $\mu = d\sigma$ via $\mu([a, b)) = \sigma(b) - \sigma(a)$

Rewrite the integral in the r. h. s. of (Sch 3.3) as

$$I = \frac{1}{2\pi i}\oint_{|z|=R}\left[\frac{2}{\zeta - z} - \frac{1}{\zeta}\right]\frac{1}{2}\left(f(\zeta) + \overline{f(\zeta)}\right)\,d\zeta\,.$$

Using Cauchy formula and expanding $f(z) = \sum a_k z^k; \overline{f(z)} = \sum \bar{a}_k \bar{z}^k$, $\frac{2}{\zeta - z} = \frac{2}{\zeta}\sum_{k=0}^{\infty}\left(\frac{z}{\zeta}\right)^k$ one gets

$$I = \frac{1}{2}\left[2f(z) - f(0)\right] +$$

$$\frac{1}{2\pi i}\oint_{|z|=R}\left\{\frac{2}{\zeta}\sum_{k=0}^{\infty}\left(\frac{z}{\zeta}\right)^k - \frac{1}{\zeta}\right\}\left[\frac{1}{2}\sum_{k=0}\bar{a}_k\bar{\zeta}^k\right]\,d\zeta =$$

$$f(z) - \frac{1}{2}f(0) + \bar{a}_0 - \frac{\bar{a}_0}{2} =$$

$$f(z) - \frac{f(0)}{2} + \frac{\overline{f(0)}}{2} = f(z) - i\Im f(0)\,.$$

HR **Theorem 2.** *(Herglotz, F. Riesz). Define the class (a. k. a. the class of pseudopositive functions)*

$$\mathcal{C}(\textit{arathéodory}) = \{f : f \ \ \textit{is} \ \ \textit{analytic} \ \ \textit{in}\ D = \{|z| < 1\}, f(D) \subset \{w : \Re w > 0\}\}\ .$$

Then

$$f \in \mathcal{C} \Longleftrightarrow f(z) = i\Im f(0) + \frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{e^{i\phi} + z}{e^{i\phi} - z}\,d\sigma(\phi)\,,$$

where σ is a non-decreasing function of bounded variation on $[-\pi, \pi]$ or, what is the same, $d\sigma$ is a positive finite Borel measure μ on $[-\pi, \pi]$: $\sigma(x) := \mu([-\pi, x]$, σ is continuous from the right, $\mathrm{Var}_a^b(\sigma) = \mu([a, b])$; the Stieltjes integral appearing in the r. h. s. is just the integral w. r. t. μ).

That is an almost immediate consequence of the Schwarz formula and Banach-Aláoglu theorem (unit ball in the dual space (space of finite regular ($\mu(F) = \inf\mu$ ("bigger open")$=\sup\mu$ ("smaller closed")) Borel measures) on $[-\pi, \pi]$ to a separable Banach space ($C([-\pi, \pi])$ is compact in weak-*-topology (see Rudin, R&C Analysis, 11.12 ans 11.19), in Stieltjes language the latter is usually replaced by (old-fashioned) Helly's theorems ("the first" and "the second").

Namely, let $f \in \mathcal{C}$. Then for $|z| < R < 1$

$$f(z) = i\Im f(0) + \frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{Re^{i\phi} + z}{Re^{i\phi} - z}\Re f(Re^{i\phi})\,d\phi$$

and

$$f(Rz) = i\Im f(0) + \frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{e^{i\phi} + z}{e^{i\phi} - z}d\left(\int_{-\pi}^{\phi}\Re f(Re^{i\theta})d\theta\right)$$

Consider the family of monotonously growing (non decreasing) functions

$$\sigma_R(\phi) := \int_{-\pi}^{\phi}\Re f(Re^{i\theta})d\theta$$

$$|z| < R < 1\,.$$

Obviously, it is uniformly bounded

$$|\sigma_R(\phi)| \le \int_{-\pi}^{\pi} \Re f(Re^{i\theta})d\theta = 2\pi R\,\Re f(0) \le 2\pi \Re f(0)$$

($\Re f$ is harmonic, mean value theorem for harmonic functions is used). Thus (Helly), $\exists R_k \to 1$ and (non decreasing) σ: $\sigma_{R_k} \to \sigma$ at the points of continuity of σ (in particular, a. e.) and

$$\int_{-\pi}^{\pi} \frac{e^{i\phi}+z}{e^{i\phi}-z} d\sigma_R(\phi) \longrightarrow \int_{-\pi}^{\pi} \frac{e^{i\phi}+z}{e^{i\phi}-z} d\sigma(\phi)$$

The most elementary proof of Theorem HR 2:

It is really striking but one could avoid any reference to Helly's and B-A's theorems. The following ingenious trick can be found in Simon's recent book on Loewner Theorem.

Lemma 7. *Let $\mathcal{F} \subset C(T)$, and let $\mathcal{F}$ is dense in $C(T)$. Let μ_r be a a family of probability measures on T and let*

$$\forall \phi \in \mathcal{F} \quad \exists \lim_{r\to 1-} \int_T \phi d\mu_r\,.$$

Then there exists a probability measure μ on T such that

$$\forall g \in C(T) \quad \lim_{r\to 1-} \int_T g d\mu_r = \int_T g\, d\mu$$

Proof. Clearly, $\forall g \in C(T)\ \exists \lim_{r\to 1-} \int_T g\, d\mu_r$:

$$\left|\int_T g\, d\mu_{r_1} - \int_T g\, d\mu_{r_1}\right| = \left|\int (g-\phi_k) d\mu_{r_1} - \int (g-\phi_k)\, d\mu_{r_2} + \int_T \phi_k d\mu_{r_1} - \int \phi_k d\mu_{r_2}\right| \le 3\frac{\epsilon}{3} = \epsilon$$

$$l(g) := \lim_{r\to 1-} g d\mu_r$$

$$|l(g)| \le ||g||_\infty$$

$$l(g) \ge 0 \ \text{ if } \quad g \ge 0$$

Now R-M Theorem implies $l(g) = \int_T g\, d\mu$.

Lemma 8. *Let $K(w,z) = \frac{w+z}{w-z}$. Then the linear span $\mathcal{F}$ of functions*

$$\theta \mapsto K(e^{i\theta}, z)$$

and

$$\theta \mapsto \overline{K(e^{i\theta}, z)}$$

with $z \in \{|z|<1\}$ is dense in $C(T)$.

Proof. One has

$$K(e^{i\theta}, z) = 1 + 2\sum_{n=1}^{\infty} z^n e^{-in\theta}$$

Derivatives $\frac{d^n}{(dz)^n} K(e^{i\theta}, z)\Big|_{z=0}$ (represented as limits) are in in the closure of the span, so all $e^{in\theta}$ $(n \geq 0)$ and there conjugates are in the closure of the span, so the statement follows from Weierstrass theorem for trigonometric polynomials.

Now applying Schwarz formula (3.3) to $\psi(z) = f(rz)$ one gets

$$f(rz) = \int K(e^{t\theta}, z) d\mu_r(\theta)$$

with $d\mu_r = \frac{1}{2\pi} \Re f(re^{i\theta}) d\theta$ and $f(rz) \to f(z)$ as $r \to 1-$ and Theorem 2 follows immediately from the two lemmas above.

Define the class $\mathcal{N}$(evanlinna):

$$\mathcal{N} = \{f \in \mathcal{A}(\{\Im z > 0\}) : f(\{\Im z > 0\}) \subset \{\Im z > 0\}\}$$

Corollary 1.

$$f \in \mathcal{N}$$

if and only if

$$f(z) = \mu z + \nu + \int_{-\infty}^{+\infty} \frac{1 + uz}{u - z} d\tau(u)$$

where $\mu, \nu \in \mathbb{R}$, $\mu > 0$ and τ is a non decreasing function of bounded variation.

Proof. From Theorem 2

$$\Phi(z) = (if(z)) = \Re\Phi(0) + \frac{i}{2\pi} \int_{-\pi}^{\pi} \frac{e^{i\theta} + z}{e^{i\theta} - z} d\sigma(\theta)$$

for $\Phi : D \to \{\Im z > 0\}$.

$$\zeta \in D \mapsto z \in \{\Im z > 0\}$$

$$\zeta = \frac{1 + iz}{1 - iz}$$

$$z = \frac{1}{i} \frac{\zeta - 1}{\zeta + 1}$$

Let $\Psi \in \mathcal{N}$, then

$$\Psi(\frac{1}{i} \frac{\zeta - 1}{\zeta + 1}) = \Phi(\zeta) = \Re\Phi(0) + \frac{i}{2\pi} \int_{-\pi}^{\pi} \frac{e^{i\theta} + \zeta}{e^{i\theta} - \zeta} d\sigma(\theta)$$

and

$$\Psi(z) = \Re\Psi(i) + \frac{i}{2\pi} \int_{-\pi}^{\pi} \frac{e^{i\theta} + \frac{1+iz}{1-iz}}{e^{i\theta} - \frac{1+iz}{1-iz}} d\sigma(\theta)$$

$$u = \tan\theta/2$$

$$e^{i\theta} = \frac{1-u^2}{1+u^2} + i\frac{2u}{1+u^2} = \frac{1+iu}{1-iu}$$

$$d\sigma(\theta) = d\tilde{\sigma}(\theta) + \mu\delta(\cdot - \pi) = d\tilde{\sigma}(2\arctan u) + \mu\delta = d\tau(u) + \mu\delta\,,$$

where $\mu > 0$ and $\tilde{\sigma}$ is left continuous at $+\pi$.

One has

$$\frac{\frac{1+iu}{1-iu} + \frac{1+iz}{1-iz}}{\frac{1+iu}{1-iu} - \frac{1+iz}{1-iz}} = \frac{1}{i}\frac{uz+1}{u-z}$$

and

$$\Psi(z) = \Re\Psi(i) + \int_{-\infty}^{+\infty} \frac{1+uz}{u-z}\frac{d\tau(u)}{2\pi} + \frac{1+uz}{u-z}\Big|_{u=+\infty}\mu$$

Introduce the class $\mathcal{R}$(esolvent):

$$\mathcal{R} = \{f \in \mathcal{N} : \sup_{y\geq 1} |yf(iy)| < \infty\}$$

Finally, we get the Herglotz Theorem as a corollary:

Corollary 2.

$$f \in \mathcal{R}$$

if and only if

$$f(z) = \int_{-\infty}^{+\infty} \frac{d\sigma(u)}{u-z}$$

with non decreasing σ of bounded variation.

Proof. From Corollary 1

$$yf(iy) = i\mu y^2 + \nu y + y\int_{-\infty}^{\infty} \frac{1+iuy}{u-iy} d\tau(u)$$

Therefore,

$$\Im(yf(iy)) = \mu y^2 + y^2 \int_{-\infty}^{\infty} \frac{1+u^2}{u^2+y^2} d\tau(y) \qquad (3.4) \quad \boxed{\texttt{Im}}$$

$$\Re(yf(iy)) = \nu y + y\int_{-\infty}^{\infty} \frac{u(1-y^2)}{u^2+y^2} d\tau(u) \qquad (3.5) \quad \boxed{\texttt{Re}}$$

and, since $f \in \mathcal{R}$,

$$|yf(iy)| \leq C \qquad (3.6) \quad \boxed{\texttt{nerav}}$$

as $y \to +\infty$. From $(\overset{\texttt{Im}}{3.4})$ and $(\overset{\texttt{nerav}}{3.6})$ one gets

$$\mu = 0 \qquad (3.7) \quad \boxed{\texttt{mu}}$$

and

$$y^2 \int_A^B \frac{1+u^2}{u^2+y^2} d\tau(y) \leq C$$

for any A, B and, therefore,

$$\int_{\mathbb{R}} (1+u^2)d\tau(u) \le C \tag{3.8}$$ $\boxed{\texttt{key}}$

From (3.5) and (3.6) one gets

$$\nu + \lim_{y\to\infty} \int_{\mathbb{R}} \frac{u(1-y^2)}{u^2+y^2} d\tau(u) = 0$$

and, therefore,

$$\nu = \int_{\mathbb{R}} u d\tau(u) \tag{3.9}$$ $\boxed{\texttt{nu}}$

Finally, Corollary 1, (3.7) and (3.9) give

$$f(z) = \int_{\mathbb{R}} \frac{(1+u^2)d\tau(u)}{u-z}$$

Introduce

$$\sigma(u) = \int_{-\infty}^{u} (1+v^2)d\tau(v)$$

(see (3.8)!). Now

$$f(z) = \int_{\mathbb{R}} \frac{d\sigma(u)}{u-z}$$

as stated in the Corollary.

Exercise. Let $f(z) = const = \beta + i\gamma$ with $\gamma > 0$. Clearly $f \in \mathcal{N}$. How Corollary 1 works here?

Answer.

$$f(z) = \beta + \frac{\gamma}{\pi} \int_{\mathbb{R}} \frac{1+uz}{u-z} \frac{du}{1+u^2}$$

(compute the integral via Cauchy theorem).

To restore the measure from $f(z)$ one uses Stiltjes-Perron formula (Akhiezer, Classical moment problem, p. 155 of Russian edition), cf. attachment to Lang's "$SL_2(\mathbb{R})$". More details will be added.

Proof of the Spectral Theorem

Let us derive the spectral theorem from the Herglotz theorem.

Herglotz $\Longrightarrow$ S. T.

End of Lecture 7

Lecture 8

Choose $f \in H$.

Then $\Phi_f(z) = (R(z;A)f, f)$ is analytic in $\mathbb{H}_+$ and has positive imaginary part there, therefore,

$$\Phi_f(z) = \int \frac{d\sigma(x)}{x - z}$$

due to the Herglotz Theorem.

Define the closed subspace, H_f, of H via

$$H_f = \overline{\mathrm{LinSpan}\Big(\{f\} \cup \{R(z;A)f : z \in \mathbb{C} \setminus \mathbb{R}\}\Big)}$$

We will construct isometric isomorphism

$$U : H_f \to L_2(\mathbb{R}, d\sigma)$$

Define

$$U(\mu f + \sum_{i=1}^{n} \lambda_i R(z_i; A)f) = \mu + \sum_{i=1}^{n} \frac{\lambda_i}{\cdot - z_i} \tag{3.10}$$ $\boxed{\texttt{isom}}$

Clearly, expression at the right gives an element of $L_2(\mathbb{R}, d\sigma)$.

Lemma 9.

$$||\mu f + \sum_{i=1}^{n} \lambda_i R(z_i; A)f; H|| = ||\mu + \sum_{i=1}^{n} \frac{\lambda_i}{\cdot - z_i}; L_2(\mathbb{R}, d\sigma)|| \,.$$

Since

$$||\sum_{i=1}^{n} v_i||^2 = \sum_{i=1}^{v} ||v_i||^2 + 2 \sum_{i<j} \Re(v_i, v_j) \,, \tag{3.11}$$ $\boxed{\texttt{Sqn}}$

it suffices to show that

1. $$(R(z,A)f, f)_H = (\frac{1}{\cdot - z}, 1)_{L_2(\mathbb{R},d\sigma)} = \int_{\mathbb{R}} \frac{d\sigma(x)}{x - z}$$

   (and that is already done $(= \Phi_f(z))$),

2. $$(R(z;A)f, R(w;A)f)_H = (\frac{1}{\cdot - z}, \frac{1}{\cdot - w})_{L_2(\mathbb{R},d\sigma)} = \int_{\mathbb{R}} \frac{d\sigma(x)}{(x - z)(x - \bar{w})}$$

   (relatively easy consequence of the resolvent identity),

3. $$(f, f)_H = (1, 1)_{L_2(\mathbb{R},d\sigma)} = \int_R d\sigma$$

   and that is the most tricky.

Let us prove 2):

$$(R(z;A)f, R(w;A)f)_H = (R^*(w;A)R(z;A)f, f) = (R(\bar{w};A)R(z;A)f, f) =$$

(by the resolvent identity)

$$\frac{1}{\bar{w}-z}([R(\bar{w};A) - R(z;A)]f, f) =$$

(using Herglotz)

$$= \frac{1}{\bar{w}-z}\int_{\mathbb{R}} \left(\frac{1}{x-\bar{w}} - \frac{1}{x-z}\right) d\sigma(x) =$$

$$\int_{\mathbb{R}} \frac{1}{x-z}\overline{\frac{1}{x-w}}\, d\sigma(x)\,.$$

Now 3):

Naive idea:

$$([I - i\epsilon A]^{-1}f, [I - i\epsilon A]^{-1}f) \sim (f, f)$$

for small ϵ. Since A is unbounded, seems wrong. In fact, works!

One has

$$[I - i\epsilon A]^{-1} = -\frac{1}{i\epsilon}R(\frac{1}{i\epsilon};A)$$

and

$$([I - i\epsilon A]^{-1}f, [I - i\epsilon A]^{-1}f) = \frac{1}{\epsilon^2}\Big(R(\frac{1}{i\epsilon};A)f, R(\frac{1}{i\epsilon};A)f\Big) =$$

due to items 1) and 2) and (3.11)

$$= \frac{1}{\epsilon^2}\int_{\mathbb{R}} \left|\frac{1}{x - \frac{1}{i\epsilon}}\right|^2 d\sigma(x) =$$

$$= \int_{\mathbb{R}} \left|\frac{1}{1 - i\epsilon x}\right|^2 d\sigma(x)$$

Now it is obvious that the right hand side has the limit ($= \int_{\mathbb{R}} d\sigma$) as $\epsilon \to 0$ (due to Lebesgue:

$$\left|\frac{1}{1 - i\epsilon x}\right|^2 \leq 1$$

and 1 is a summable majorant ($d\sigma$ is a finite measure!))

What about the left hand side? It turns out that $[I - i\epsilon A]^{-1}f]$ converges:

$$||[I - i\epsilon_1 A]^{-1}f - [I - i\epsilon_2 A]^{-1}f||^2 =$$

due to 1), 2) and (3.11)

$$= \int_{\mathbb{R}} \left|\frac{1}{i\epsilon_1}\frac{1}{x - \frac{1}{i\epsilon_1}} - \frac{1}{i\epsilon_2}\frac{1}{x - \frac{1}{i\epsilon_2}}\right|^2 d\sigma(x) \to 0$$

as $\epsilon_1, \epsilon_2 \to 0$ (due to Lebesgue Theorem). Thus,

$$g_\epsilon := [I - i\epsilon A]^{-1} f \to g$$

with some $g \in H$ as $\epsilon \to 0$.

One has

$$[I - i\epsilon A] g_\epsilon = f$$

and

$$A(i\epsilon g_\epsilon) = g_\epsilon - f \to g - f\,.$$

But $i\epsilon g_\epsilon \to 0$ as $\epsilon \to 0$. Thus, $[0, g-f] \in \Gamma(\bar{A})$. Since A is closed, $g = f$, and

$$(f,f)_H = \int_R d\sigma\,.$$

as was stated.

Lemma 10. *The range of U is dense:*

$$\overline{R(U)} = L_2(\mathbb{R}; d\sigma)\,.$$

Proof. Clearly, $R(U)$ contains constants and fractions $\frac{c}{\cdot - z}$ with $z \in \mathbb{C} \setminus \mathbb{R}$. Moreover, since

$$\frac{1}{(\cdot - z)^n} = \lim_{\epsilon \to 0} \frac{1}{(\cdot - z)(\cdot - z + \epsilon) \ldots (\cdot - z + (n-1)\epsilon)} =$$

$$= \lim_{\epsilon \to 0} \sum_{k=0}^{n-1} \frac{A_k}{\cdot - z + k\epsilon}$$

(lim is taken in $L_2(\mathbb{R}; d\sigma)$), the closure $\overline{R(U)}$ contains contains all rational functions

$$\frac{p_m(x)}{q_n(x)} = c + \sum \frac{A_{k,l}}{(x - z_k)^l}$$

with $m \le n$ and poles lying outside $\mathbb{R}$.

Clearly, the set of such rational functions forms an *-algebra (check!) of continuous functions on one-point compactification, $\bar{\mathbb{R}}$, of $\mathbb{R}$ (one can think about this object as S^1) which separates the points of $\bar{\mathbb{R}}$ and vanish at no point of $\bar{\mathbb{R}}$. Due to Stone-Weierstrass theorem this set is dense in $C(\bar{\mathbb{R}})$ (in uniform norm) and, therefore, any function from $C(\bar{\mathbb{R}})$ can be approximated by such rational functions in $L_2(R, d\sigma)$-norm. In particular,

$$\overline{R(U)} \supset C_{comp}(\mathbb{R})\,.$$

Let $f \perp \overline{R(U)}$. Then

$$\int_{\mathbb{R}} m\bar{f} d\sigma = 0$$

for any $m \in C_{comp}(\mathbb{R})$. Since $f \in L_2(\mathbb{R}; d\sigma)$ and $d\sigma$ is a finite measure, $f \in L_1(\mathbb{R}; d\sigma)$ and $\bar{f} d\sigma$ is a (comlex-valued) measure $d\mu$:

$$\int_{\mathbb{R}} g d\mu = \int_{\mathbb{R}} g \bar{f} d\sigma$$

Now

$$\forall g \in C_{comp} \quad (\mathbb{R}) \int g d\mu = 0 \implies \mu = 0$$

and, therefore, $f = 0$ a. e. with respect to $d\sigma$. Thus $f = 0$ as an element in $L_2(\mathbb{R}, d\sigma)$ and the lemma is proved.

These two lemmas immediately imply that

Proposition 18. *The map (3.10) extends to isometric isomorphism*

$$U : H_f \to L_2(\mathbb{R}, d\sigma)$$

Lemma 11. *For any $z \in \mathbb{C} \setminus \mathbb{R}$*

$$R(z; A) H_f \subset H_f \, .$$

Proof. For $z \neq z_k$ we have from the resolvent identity:

$$R(z; A) R(z_k; A) f = \frac{1}{z - z_k} \left(R(z; A) f - R(z_k; A) f \right) \in H_f \, .$$

For $z = z_k$ one uses Proposition 15 (derivative of the resolvent)

$$R(z; A) R(z; A) f = \frac{d}{dz} R(z; A) f = \lim_{w \to z} \frac{1}{z - w} \left(R(z; A) f - R(w; A) f \right) \in H_f \, .$$

Crucial fact

Resolvent operator, $g \mapsto R(z; A) g$, in H_f is unitary equivalent to the operator of the multiplication,

$$\phi(x) \mapsto \frac{1}{x - z} \phi(x) \, ,$$

by the function $\frac{1}{\cdot - z}$ in $L_2(\mathbb{R}, d\sigma)$.

Lemma 12. *Let $z \in \mathbb{C} \setminus \mathbb{R}$. Then*

$$U R(z; A) = \frac{1}{\cdot - z} U \tag{3.12}$$ rerepr

Proof. It suffices to check (3.12) on a dense set.

$$U R(z; A) \big(\mu f + \sum \lambda_k R(z_k; A) f \big) = U \big(\mu R(z; A) f + \sum \frac{\lambda_k}{z - z_k} (R(z; A) f - R(z_k; A) f) \big) =$$

$$= \frac{\mu}{\cdot - z} + \sum \frac{\lambda_k}{z - z_k} \left(\frac{1}{\cdot - z} - \frac{1}{\cdot - z_k} \right) =$$

$$= \frac{1}{\cdot - z} \big(\mu + \sum \frac{\lambda_k}{\cdot - z_k} \big) =$$

$$= \frac{1}{\cdot - z} U \big(\mu f + \sum \lambda_k R(z_k; A) f \big) \, .$$

End of Lecture 8

Lecture 9

End of the proof of the Spectral Theorem

Step 1:

$$H = \oplus_{k=1}^{\infty} H_{f_k}$$

(Of course, the sum can be finite.)

Let $g_1, g_2, \ldots$ be a ONB in H.

$$f_1 := g_1$$

$$H = H_{f_1} \oplus H_{f_1}^{\perp}$$

Clearly, the orthogonal complement, $H_{f_1}^{\perp}$ is invariant w. r. t. all operators $R(z;A)$ with $z \in \mathbb{C} \setminus \mathbb{R}$:

In fact, let $g \in H_{f_1}^{\perp}$ then for any $h \in H_f$

$$(h, R(z;A)g) = (R(\bar{z};A)h, g) = 0$$

due to Lemma 7, and, therefore, $R(z;A)g \in H_{f_1}^{\perp}$.

Take the minimal integer k such that

$$\mathrm{Proj}_{H_{f_1}^{\perp}} g_k \neq 0$$

and put

$$f_2 := \mathrm{Proj}_{H_{f_1}^{\perp}} g_k \in H_{f_1}^{\perp}$$

Then $H_{f_2} \perp H_{f_1}$.

Clearly, $(H_{f_1} \oplus H_{f_2})^{\perp}$ is invariant w. r. t. to all $R(z;A)$ as the intersection of two invariant subspaces:

$$(H_{f_1} \oplus H_{f_2})^{\perp} = H_{f_1}^{\perp} \cap H_{f_2}^{\perp}$$

Let l be the smallest integer such that

$$\mathrm{Proj}_{(H_{f_1} \oplus H_{f_2})^{\perp}} g_l \neq 0$$

Take

$$f_3 := \mathrm{Proj}_{(H_{f_1} \oplus H_{f_2})^{\perp}} g_l$$

and so on.

Step 2: From Step 1, H is isometric to

$$\oplus_{k=1}^{\infty} L_2(\mathbb{R}^{(k)}; d\sigma^{(k)})$$

and $R(z;A)$ in H is unitary equivalent to multiplication by $\frac{1}{\cdot - z}$ in each $L_2(\mathbb{R}^{(k)}; d\sigma^{(k)})$.

What happens to $D(A)$ under this isometry?

Take any $z \in \mathbb{C} \setminus \mathbb{R}$. Then

$$D(A) = \{h \in H : \exists h_1 \in H \text{ such that } h = R(z;A)h_1\}$$

$(A - zI : D(A) \to H$ and $R(A - zI) = H$ - see the statement after Lemma 3, page 30)
Thus $D(A)$ turns into

$$\{\psi \in L_2 : \exists \phi \in L_2 \text{ such that } \psi = \frac{1}{\cdot - z}\phi\}$$

or, what is the same

$$\{\psi \in L_2 : (\cdot - z)\psi \in L_2\}$$

or, what is the same

$$\{\psi \in L_2; x\psi \in L_2\}$$

!!!

On the other hand

$$A = R(z;A)^{-1} + zI : D(A) \to H$$

and that is multiplication $g(x) \mapsto xg(x)$ in $\oplus_{k=1}^{\infty} L_2(\mathbb{R}^{(k)}; d\sigma^{(k)})$. **Spectral Theorem is proved.**

4 Kato-Rellich and around

Definition 10. *Let $\alpha \in \mathbb{R}$. Operator B is called α-bounded with respect to operator A if*

- $D(B) \supset D(A)$.
- $\forall x \in D(A)$ *one has*

$$||Bx|| \leq \alpha ||Ax|| + C||x|| \, .$$

Remark. Clearly,

$$||Bx||^2 \leq \alpha^2 ||Ax||^2 + C^2 ||x||^2 \implies ||Bx|| \leq \alpha ||Ax|| + C||x|| \, .$$

and, since

$$2\alpha C ||Ax|| \, ||x|| = 2\delta\alpha ||Ax|| \frac{C}{\delta} ||x|| \leq \delta^2 \alpha^2 ||Ax|^2 + \left(\frac{C}{\delta}\right)^2 ||x||^2 \, ,$$

one has

$$||Bx|| \leq \alpha ||Ax|| + C||x|| \implies ||Bx||^2 \leq (\alpha + \epsilon)^2 ||Ax||^2 + C_1 ||x||^2$$

Proposition 19. *Let B is α-bounded w. r. t. A and $\alpha < 1$. Then*

$$A \text{ is closed} \implies A + B \text{ is closed.}$$

Reminder: one has to prove that

$$\boxed{x_n \to x_0; \quad (A+B)x_n \to y_0 \implies x_0 \in D(A+B) \;\; \& \;\; (A+B)x_0 = y_0}$$

Proof. Straightforward:

$$||Ax_n - Ax_m|| \le ||(A+B)(x_n - x_m) - B(x_n - x_m)|| \le ||(A+B)(x_n - x_m)|| + ||B(x_n - x_m)|| \le$$

$$||(A+B)(x_n - x_m)|| + \alpha||A(x_n - x_m)|| + C||x_n - x_m||\,.$$

Since $\alpha < 1$ this implies

$$||Ax_n - Ax_m|| \le C_1(||(A+B)(x_n - x_m)|| + ||x_n - x_m||)$$

Since A is closed, $x_0 \in D(A)$ and $Ax_n \to z = Ax_0$. On the other hand, since

$$||B(x_n - x_0)|| \le \alpha||Ax_n - Ax_0|| + C||x_n - x_0||\,,$$

$$Bx_n \to Bx_0\,.$$

Thus

$$(A+B)x_n \to (A+B)x_0 = y_0\,.$$

Proposition 20. *Let A be self-adjoint and let B be symmetric. Assume that B is α-bounded w. r. t. A and $\alpha < 1$. Then $A+B$ is self-adjoint.*

We will be using criterion from page 11:

$$\boxed{\begin{array}{c} B - \text{densely defined, symmetric} \\ \\ (\exists \lambda \in \mathbb{C} : R(B - \lambda I) = R(B - \bar{\lambda} I) = H) \implies B \;\; \text{is} \;\; \text{self} - \text{adjoint} \end{array}}$$

We will prove that

$$R(A + B \pm icI) = H$$

for any $c > C$, where C is the constant from the estimate

$$||Bx||^2 \le \alpha^2||Ax||^2 + C^2||x||^2\,. \tag{4.1}$$ est2

Step 1. $R(A + B \pm icI)$ is a closed subspace of H.

This is true in more general case (since $A+B$ is closed due to the previous proposition and symmetric):

$$\boxed{T \text{ - closed, symmetric} \implies R(T - zI) \text{ is closed for any } z \in \mathbb{C} \setminus \mathbb{R}.}$$

As in (est1 3.2) for $u \in D(T)$ one has the inequality

$$||(T - z)u||^2 \ge |\Im z|^2||u||^2.$$

Let $x_n \in R(T - zI)$, $x_n \to x_0$. Then $x_n = (T - zI)y_n$ for $y_n \in D(T)$ and

$$||x_n - x_m||^2 = ||(T - z)(y_n - y_m)||^2 \ge |\Im z|^2||y_n - y_m||^2$$

Thus, $y_n \to y_0$ and, since $T - zI$ is a closed operator, $y_0 \in D(T)$ and $(T - zI)y_0 = x_0$. Thus $x_0 \in R(T - zI)$.

Step 2. $R(A + B + icI) = H$.

Let $h \perp R(A + B + icI)$. We have to show that $h = 0$. One has

$$((A + B + icI)x, h) = 0$$

for all $x \in D(A)$. (Reminder: $D(B) \supset D(A)$.)

The operator A is self-adjoint, therefore, spectrum $(A) \subset \mathbb{R}$ and, therefore,

$$R(A + icI) = H\,.$$

Therefore,

$$h = (A + icI)y$$

with some $y \in D(A)$ and

$$((A + B + icI)x, (A + icI)y) = 0$$

for all $x \in D(A)$. Let

$$x := y\,.$$

Thus,

$$((A + B + icI)y, (A + icI)y) = 0\,.$$

This gives

$$||(A + icI)y||^2 + (By, A + icI)y) = 0\,.$$

Now follows simple estimate, using (est2 4.1):

$$||(A + icI)y||^2 = |(By, A + icI)y)| \leq ||By||\,||(A + icI)y||$$

implies

$$||(A + icI)y|| \leq ||By||$$

and

$$||(A + icI)y||^2 \leq ||By||^2 \leq \alpha^2 ||Ay||^2 + C^2 ||y||^2$$

But, due to symmetry of A,

$$||A + icI)y||^2 = ||Ay||^2 + c^2 ||y||^2\,.$$

Therefore,

$$(c^2 - C^2)||y||^2 \leq (\alpha^2 - 1)||Ay||^2 \leq 0$$

Since $c > C$, this gives $y = 0$ and, therefore, $h = 0$.

The same works for $A + B - icI$, therefore, $A + B$ is self-adjoint.

Self-adjoint Laplacian in $\mathbb{R}^n$ $(n = 3)$

Consider the Laplace operator

$$-\Delta = -\sum_{n=1}^{3} \frac{\partial^2}{\partial^2 x_n}\,.$$

Let $F : L_2(\mathbb{R}^3) \to L_2(\mathbb{R}^3)$ be the Fourier transform (reminder: it is a unitary operator):

$$(Fu)(y) = \frac{1}{(2\pi)^{3/2}} \int_{\mathbb{R}^3} e^{-ixy} u(x)\,dx =: \hat{u}(y)$$

One has

$$F(\frac{1}{i}\partial_{x_k} u) = y_k F(u) = y_k \hat{u}(y)$$

(cf. $\frac{1}{i}\frac{d}{dx}$ in Section 2.1)

$$F(-\Delta u) = (\sum_{k=1}^{3} y_k^2) Fu$$

$$-\Delta_z u(z) = F^{-1}_{y\to z} \sum_{k=1}^{3} y_k^2 F_{x\to y} u(x)$$

Let $\tilde{A}$ be the self-adjoint operator of multiplication by $\sum_{k=1}^3 y_k^2$ in $L_2(\mathbb{R}^3)$.

$$D(\tilde{A}) = \{v \in L_2(\mathbb{R}^3) : (\sum_{k=1}^{3} y_k^2) v \in L_2\}$$

Then $A := F^{-1}\tilde{A}F (=: -\Delta)$ is self-adjoint with

$$D(A) = W_2^2(\mathbb{R}^3) = H^2(\mathbb{R}^3)$$

$$||u; H^2(\mathbb{R}^3)||^2 = \int_{\mathbb{R}^3} |\hat{u}(y)|^2 (1 + |y|^2)^2 dy$$

$C_0^\infty(\mathbb{R}^3)$ is dense in $H^2(\mathbb{R}^3)$

Theorem 3. *Let $V = V_1 + V_2$, where $V_1 \in L_2(\mathbb{R}^3)$ and $V_2 \in L_\infty(\mathbb{R}^3)$. Then the operator*

$$\mathcal{H} = -\Delta + V(x)$$

is self-adjoint ($\mathcal{H} : L_2(\mathbb{R}^3) \to L_2(\mathbb{R}^3)$; $D(\mathcal{H}) = H^2(\mathbb{R}^3)$).

End of Lecture 9

Lecture 10

Proof. We are to prove that the operator (of multiplication by) $V_1 + V_2$ is α-bounded w. r. t. $-\Delta$ with $\alpha < 1$.

The term $V_2 \in L_\infty$ is of no interest, since

$$||V_2 u||_2 \le ||V_2||_\infty ||u||_2 \le C||u||_2\,.$$

(Here $||\cdot||_2$ is the L_2-norm, $||\cdot||_\infty$ is the L_∞-norm.)

Thus, let us estimate the norm of $||V_1 u||$ for $u \in D(-\Delta) = H^2(\mathbb{R}^3)$.

Let $f(= V_2) \in L_2$, $u \in C_0^\infty(\mathbb{R}^3)$ (which is dense in H^2).

We have

$$||fu||_2 \leq ||f||_2 ||u||_\infty \leq \frac{1}{(2\pi)^{3/2}} ||f||_2 ||\hat{u}||_1$$

Reminder:

$$u = F^{-1}\hat{u} = \frac{1}{(2\pi)^{3/2}} \int_{\mathbb{R}^3} \hat{u}(\xi) e^{i\xi x} d\xi$$

$$|u(x)| \leq \frac{1}{(2\pi)^{3/2}} \int_{\mathbb{R}^3} |\hat{u}(\xi)|\, d\xi$$

Moreover,

$$||\hat{u}||_1 = ||(1+|y|^2)^{-\delta}(1+|y|^2)^{\delta}\hat{u}||_1 \leq$$

(Cauchy inequality)

$$\leq ||(1+||y|^2)^{-\delta}||_2 ||(1+|y|^2)^{\delta}\hat{u}||_2$$

for any δ such that the integral

$$\int_{\mathbb{R}^3} \frac{dy}{(1+|y|^2)^{2\delta}}$$

converges, that is for $2\delta > 3/2$ or $\delta > 3/4$ (and $3/4 < 1$!!!!).

Therefore,

$$||\hat{u}||_1 \leq C||(1+|y|^2)^{\delta}\hat{u}||_2$$

with some δ between 3/4 and 1 (strictly less than one!!).

Now follows *the central point:*

Clearly, for $0 < \delta < 1$

$$x^\delta \leq \epsilon x + C(\epsilon)$$

for any $\epsilon > 0$. (Make a picture: the graphs of x^δ and ϵx. Then move the second graph up, adding C.)

Therefore,

$$(1+|y|^2)^\delta \leq \epsilon(1+|y|^2) + C(\epsilon)$$

and

$$||\hat{u}||_1 \leq C||(\epsilon(1+|y|^2) + C(\epsilon))\hat{u}||_2 \leq \epsilon C||\,|y|^2\hat{u}||_2 + C\,C(\epsilon)||u||_2 \leq$$

$$\leq \epsilon C|| - \Delta u||_2 + C_1(\epsilon)||u||_2$$

Finally,

$$||(V_1 + V_2)u||_2 \leq C\epsilon|| - \Delta u||_2 + C(\epsilon)||u||_2\,,$$

where $C\epsilon < 1$ (!!!) for sufficiently small ϵ.

Main application:

$$\boxed{-\Delta - \frac{e^2}{r}}$$

with domain $H^2(\mathbb{R}^3)$ is self-adjoint.

$$\frac{1}{r} = \chi_{\{|y|\le 1\}}\frac{1}{r} + (1-\chi_{\{|y|\le 1\}})\frac{1}{r} = V_1 + V_2$$

with $V_1 \in L_2(\mathbb{R}^3)$, $V_2 \in L_\infty(\mathbb{R}^3)$.

Information: With slightly more effort (the same ideas!) one can prove Kato's famous result:

$(x_1, \ldots, x_n) \in \mathbb{R}^{3n}$, $x_k \in \mathbb{R}^3$. The operator

$$-\sum_{k=1}^{n} \Delta_{x_k} - \sum_{k=1}^{n} \frac{ne^2}{|x_k|} + \sum_{k<l}^{n} \frac{e^2}{|x_k - x_l|}$$

in $L_2(\mathbb{R}^{3n})$ is self-adjoint. (Atom with n electrons and nucleus of the charge $+ne$; more precise: essentially self-adjoint on $C_0^\infty(\mathbb{R}^{3n})$. See Theorem X.16 in the Reed-Simon book (easy reading with your present knowledge).)

5 Von Neumann Theory of Self-adjoint Extensions

5.1 Deficiency indices of a symmetric operator

Let A be a *closed* operator and let

$$\forall x \in D(A) \quad ||Ax|| \ge c||x||$$

with some $c > 0$. Then as in Step 1 of Proposition 20 on p. 40 one proves that $R(A)$ is a closed subspace of H.

Consider an operator B with $D(B) \supset D(A)$ subject to the estimate

$$||Bx|| \le \alpha||Ax||$$

for any $x \in D(A)$ with $\alpha < 1$.

Proposition 21. *1. $R(A+B)$ is a closed subspace of H,*

2. $\dim R(A+B)^\perp = \dim R(A)^\perp$

(or, what is the same, $\operatorname{codim} R(A+B) = \operatorname{codim} R(A)$*).*

Proof.

1. B is α-bounded w. r. t. A, $\alpha < 1$, A is closed, therefore, $A+B$ is closed (Prop. 19). On the other hand

$$||(A+B)x|| \ge ||Ax|| - ||Bx|| \ge (1-\alpha)||Ax||.$$

This proves that $R(A+B)$ is closed.

2. Let

$$H_1 = R(A+B)^\perp; \quad H_2 = R(A)^\perp.$$

Lemma 13. *Let H_1, H_2 be finite-dimensional subspaces of H and let*

$$\dim H_1 > \dim H_2.$$

Then there exists $f \in H_1$, $f \neq 0$ such that $f \perp H_2$.

(Clearly, in the case $\dim H_1 = \infty$ and $\dim H_2 < \infty$ the statement is also true).

Proof. Simple linear algebra: if $m < n$ then a homogeneous system

$$\sum_{k=1}^{n} (f_k, g_l) x_k = 0; \quad l = 1, 2, \dots m$$

(where $\{f_k\}$ is a basis in H_1; $\{g_l\}$ is a basis in H_2) with m equations and n unknowns always has a non-trivial solution.

Now assume that $\dim H_2 > \dim H_1$.

Using lemma, consider $f \in H_2 = R(A)^\perp$, $f \neq 0$ such that $f \perp H_1 = R(A+B)^\perp$. Since $R(A+B)$ is closed, $f \in R(A+B)$, therefore,

$$f = (A+B)y$$

with $y \in D(A)$. In particular (since $f = (A+B)y \in R(A)^\perp$),

$$(Ay, (A+B)y) = 0$$

and

$$0 = ||Ay||^2 + (Ay, By) \geq ||Ay||^2 - ||Ay|| \, ||By|| \geq (1-\alpha)||Ay||^2 \geq c(1-\alpha)||y||^2$$

Thus, $y = 0$ and, therefore, $f = 0$ which gives a contradiction.

Assume $\dim H_2 < \dim H_1$.

Using lemma, take $g \in R(A+B)^\perp$, $g \neq 0$ such that $g \perp R(A)^\perp$. Then $g \in R(A)$, $g = Ax$ for some $x \in D(A)$. Thus

$$((A+B)x, Ax) = 0$$

and $x = 0$ as before. Thus $g = 0$ and we get a contradiction.

Immediate application to symmetric operators:

Let A be a closed symmetric operator. Then (as in 4.1; symmetry suffices there, no need to require self-adjointness)

$$||(A - zI)x|| \geq |\Im z| \, ||x|| \, .$$

Therefore, $R(A - z)$ is a closed subspace of H for $\Im z \neq 0$.

Define

$$\beta_z := \dim R(A - zI)^\perp \, .$$

Now it is clear that

$$\beta_z \text{ is constant in the upper and the lower half-planes.}$$

Proposition 22.
$$\beta_z|_{\Im z>0} = \text{const}_1; \quad \beta_z|_{\Im z<0} = \text{const}_2$$

In fact
$$A - zI = (A - z_0 I) + (z - z_0)I$$
$$||(A - z_0 I)x|| \geq |\Im z_0|\,||x||$$

Thus, consider
$$\mathcal{A} := A - z_0 I,$$
$$\mathcal{B} := (z - z_0)I$$

and make use of Proposition 21.

One has
$$||\mathcal{B}x|| = ||(z - z_0)x|| = |z - z_0|\,||x|| \leq \frac{|z - z_0|}{|\Im z|}||(A - z_0)x|| \leq \alpha||\mathcal{A}x||$$

for
$$|z - z_0| < \alpha|\Im z_0|\,.$$

The rest is standard. Take two points in $\mathbb{H}_+$ and a contour γ connecting them and lying in $\mathbb{H}_+$, γ is compact and covered by disks, take finite sub-covering, etc.

Now one can introduce the following definition.

> A closed symmetric operator
> $$n_+ := \dim R(A - \lambda I)^\perp; \quad \Im\lambda < 0$$
> $$n_- := \dim R(A - \lambda I)^\perp; \quad \Im\lambda > 0$$
> $$0 \leq m, n \leq +\infty$$

Definition 11. *(n_+, n_-) are called deficiency indices of a symmetric closed operator A.*

> Reminder
> $$\overline{D(T)} = H$$
> $$H = \overline{R(T)} \oplus \operatorname{Ker} T^*$$

Proposition 23. *1. $A = A^* \implies n_- = n_+ = 0$*

2. If A is a closed symmetric operator then
$$n_+ = n_- = 0 \implies A = A^*$$

End of Lecture 10

Lecture 11

Proof. 1. One has

$$H = \overline{R(A-iI)} \oplus \mathrm{Ker}\,(A-iI)^* = \overline{R(A-iI)} \oplus \mathrm{Ker}\,(A+iI)\,.$$

On the other hand $R(A+iI)$ is closed and $\mathrm{Ker}\,(A+iI) = 0$ ($||(A+iI)X|| \geq ||x||$). Therefore, $R(A-iI) = H$ and $n_- = 0$.

Similarly, $R(A+iI) = H$ and $n_+ = 0$.

2. One has $R(A+iI) = R(A-iI) = H$ (since both are closed with zero orthogonal complement; closeness of A matters!). Criterion of self-adjointness (page 11) implies $A = A^*$.

Once again:

$$\boxed{\begin{array}{c} A \text{ symmetric, closed (densely defined)} \\ \\ n_+ = n_- = 0 \Leftrightarrow A = A^* \end{array}}$$

Remark 1.

$$n_+ = \dim R(A-\lambda I)^\perp = \dim \mathrm{Ker}\,(A^* - \bar{\lambda} I); \quad \Im\lambda < 0$$

$$n_- = \dim R(A-\lambda I)^\perp = \dim \mathrm{Ker}\,(A^* - \bar{\lambda} I); \quad \Im\lambda > 0$$

5.2 Self-adjoint extensions of symmetric operators

From now on A is a closed symmetric operator.

Reminder 1. $\overline{D(T)} = H$; $H = \overline{R(T)} \oplus \mathrm{Ker}\,T^*$:

$$y \in R(T)^\perp \iff \forall x \in D(T) \ \ (Tx, y) = 0 \iff y \in D(T^*) \text{ and } T^*y = 0 \iff y \in \mathrm{Ker}\,T^*$$

Reminder 2.

$$n_+ = \dim R(A+iI)^\perp = \dim \mathrm{Ker}\,(A^* - iI)$$

$$n_- = \dim R(A-iI)^\perp = \dim \mathrm{Ker}\,(A^* + iI)$$

($\dim R(A-\lambda I)^\perp$ is independent of λ with $\Im\lambda < 0$ and coincides with n_+. Take $\lambda = -i$, etc.)

Definition 12. *Introduce the closed subspaces*

$$\boxed{\begin{array}{c} \mathcal{K}_+ := R(A+iI)^\perp = \mathrm{Ker}\,(A^* - iI) \subset D(A^*) \\ \mathcal{K}_- := R(A-iI)^\perp = \mathrm{Ker}\,(A^* + iI) \subset D(A^*) \end{array}}$$

Clearly,

$$\forall x \in \mathcal{K}_\pm \quad A^*x = \pm ix$$

Proposition 24. *Let A_1 be a closed and symmetric operator such that*

$$A \subset A_1\,.$$

Then

$$A_1 \subset A^*$$

Proof.

$$A_1 \supset A \implies A_1^* \subset A^* .$$

But A_1 is symmetric, so $A_1 \subset A_1^*$. Therefore, $A_1 \subset A^*$.

Thus,

> **Domains of all symmetric extensions of A are subspaces of $D(A^*)$.**
> **A^* extends any symmetric extension of A.**
> Therefore, $D(A^*)$ is often called *the maximal domain.*

Reminder 3. A^* is not symmetric if A is not self-adjoint.
(Clear: If A^* is symmetric then $A^* \subset (A^*)^* = \bar{A} = A \subset A^*$ and $A = A^*$)

We are going to describe all closed symmetric extensions of A. (If $n_+ = n_- = 0$ then $A = A^*$ and there are no non-trivial closed symmetric extensions of A due to Proposition 24.)

5.2.1 A-orthogonality

Define new hermitian product on $D(A^*)$:

Let $x, y \in D(A^*)$

$$\boxed{\langle\langle x, y \rangle\rangle_A := (x, y) + (A^*x, A^*y)}$$

(cf. Section 1.2; Since A^* is closed, $D(A^*)$ is a Hilbert space w. r. t. $\langle\langle \cdot, \cdot \rangle\rangle$)

Thus, we have the notions of A-orthogonality and A-closedness. (It is better to say A^*-orthogonality, of course. In what follows the index A will be often omitted.)

$$x \perp_A y = 0 \iff \langle\langle x, y \rangle\rangle = 0$$

$\mathfrak{M} \subset D(A^*)$ is A-closed $\iff \overline{\mathfrak{M}}^A = \mathfrak{M}$ (closure is taken in A-norm)

Moreover, introduce the sesquilinear form on $D(A^*)$:

$$\boxed{[x, y] = (A^*x, y) - (x, A^*y)}$$

Definition 13. *A linear subspace L of $D(A^*)$ is called A-symmetric if*

$$\forall x, y \in L \quad [x, y] = 0$$

Example. $L = D(A)$ is A-symmetric.
(Since A is symmetric, $A \subset A^*$)

MAIN OBSERVATION (tautological)

$$A \subset A_1$$

$$A_1 \text{ is symmetric and closed}$$

$$\text{if and only if}$$

$$A_1 = A^*\Big|_{\mathcal{L}},$$

where $\mathcal{L}$ is A-closed and A-symmetric subspace of $D(A^*)$ containing $D(A)$ $(D(A^*) > \mathcal{L} > D(A))$.

We are to describe all A-closed and A-symmetric subspaces of $D(A^*)$ containing $D(A)$.

5.2.2 The First John von Neumann Formula

Once again:

$$\mathcal{K}_+ := \mathrm{Ker}\,(A^* - iI) \subset D(A^*)$$
$$\mathcal{K}_- := \mathrm{Ker}\,(A^* + iI) \subset D(A^*)$$

Proposition 25. $D(A)$, $\mathcal{K}_\pm$ *are A-closed and*

$$D(A^*) = D(A) \oplus_A \mathcal{K}_- \oplus_A \mathcal{K}_+ \tag{5.1}$$ JvN1

Proof. Since A is a closed operator and $A \subset A^*$, $D(A)$ is A-closed. Since $\mathcal{K}_\pm$ are closed in the ordinary H-norm (as orthogonal complements to $\overline{R(A \pm iI)}$), they are closed in A-norm (explain!).

Step 1. The sum in (5.1) is in fact A-orthogonal:

Let $x \in D(A)$ and $y \in \mathcal{K}_+$. Then

$$\langle\!\langle x, y \rangle\!\rangle_A = (x, y) + (A^*x, A^*y) = (x, y) + (A^*x, iy) =$$

$$= (x, y) + (Ax, iy) = (x - iAx, y) = -i(ix + Ax, y) = 0$$

since $\mathcal{K}_+ = R(A + iI)^\perp$.

Similarly $\langle\!\langle x, y \rangle\!\rangle_A = 0$ if $x \in D(A)$ and $y \in \mathcal{K}_-$.

Let $x \in \mathcal{K}_+$ and $y \in \mathcal{K}_-$. Then

$$\langle\!\langle x, y \rangle\!\rangle_A = (x, y) + (A^*x, A * y) = (x, y) + (ix, -iy) = 0\,.$$

Step 2. The sum is the whole $D(A^*)$. Let $h \in D(A^*)$ be A-orthogonal to the sum at the right hand side of (5.1).

Then $h \perp_A D(A)$:

$\forall x \in D(A)$

$$(x, h) + (A^*x, A^*h) = 0$$

Since $Ax = A^*x$ this gives

$$(Ax, A^*h) = -(x, h)$$

and, therefore, $A^*h \in D(A^*)$ and

$$A^*A^*h = -h$$

which is equivalent to

$$(A^* + iI)(A^* - iI)h = 0$$

which implies

$$(A^* - iI)h \in \mathrm{Ker}\,(A^* + iI) = \mathcal{K}_-$$

We will show that

$$(A^* - iI)h = 0\,. \tag{5.2}$$ kplus

That will imply that $h \in \mathcal{K}_+$. Since h is A-orthogonal to the r. h. s. of (5.1), this gives $h = 0$ what is needed.

It suffices to show that $\forall y \in \mathcal{K}_-$

$$(y, (A^* - iI)h) = 0$$

($\mathcal{K}_-$ is closed in H-norm!!! So, here stands hermitian product in H!)

In fact, we have

$$(y, (A^* - iI)h) = (y, A^*h) + i(y, h) =$$

(use $-iy = A^*y$!)

$$= (iA^*y, A^*h) + i(y, h) = i\langle\!\langle y, h\rangle\!\rangle_A = 0$$

(since h is A-orthogonal to the r. h. s. of (5.1).

Informally: Due to "JvN-I" (= (5.1), A-symmetric A-closed subspaces of $D(A^*)$ containing $D(A)$ are completely defined by their "$\mathcal{K}_+ \oplus_A \mathcal{K}_-$"-part.

Formally:

Proposition 26. *Let*

$$\mathfrak{N} = \{S < \mathcal{K}_+ \oplus_A \mathcal{K}_- \ : \ S \ \text{ is } \ A-\text{closed} \quad \text{and} \quad A-\text{symmetric} \ \}$$

$$\mathfrak{M} = \{\mathcal{L} < D(A^*) : \ \mathcal{L} \supset D(A), \ \mathcal{L} \ \text{ is } \ A-\text{closed} \quad \text{and} \quad A-\text{symmetric} \ \}$$

Define the map

$$\mathbf{map} : \mathfrak{N} \to \mathfrak{M}$$

via

$$\mathbf{map}(S) = D(A) \oplus_A S\,.$$

Then this map is correctly defined and is one to one.

Proof. The only thing that is not immediate:

$D(A) \oplus_A S$ is A-symmetric if S is A-symmetric.
Take two elements, $\phi + \phi_1$ and $\psi + \psi_1$ of $D(A) \oplus_A S$ with $\phi, \psi \in D(A)$ and $\phi_1, \psi_1 \in S$.
Then

$$[\phi + \phi_1, \psi + \psi_1] = [\phi, \psi] + [\phi_1, \psi_1] + [\phi, \psi_1] + [\psi, \phi_1]$$

and the first two terms at the right are equal to zero. On the other hand

$$[\phi, \psi_1] = (A^*\phi, \psi_1) - (\phi, A^*\psi_1) = (A\phi, \psi_1) - (\phi, A^*\psi_1) = 0$$

because $\psi_1 \in D(A^*)$. Similarly, $[\psi, \phi_1] = 0$. Thus,

$$[\phi + \phi_1, \psi + \psi_1] = 0$$

and $D(A) \oplus_A S$ is A-symmetric.

(Just in case it is not clear:

Back:

Let $\mathcal{L} \supset D(A)$, $\mathcal{L}$ - A-closed and A-symmetric. Take

$$S := \mathcal{L} \cap (\mathcal{K}_- \oplus_A \mathcal{K}_+)$$

Let $\phi \in \mathcal{L}$. Then

$$\phi = \phi_0 + \phi_1$$

with $\phi_0 \in D(A) \subset \mathcal{L}$ and $\phi_1 \in \mathcal{K}_- \oplus_A \mathcal{K}_+$. This implies $\phi_1 \in \mathcal{L}$ and, therefore, $\phi_1 \in S$. Thus,

$$\mathcal{L} = D(A) \oplus_A S\,.)$$

5.3 The Second John von Neumann Formula

We deal only with case $n_-, n_+ < +\infty$.

We will show that there is **one to one correspondence** between

- Closed symmetric extensions $A_1 (\supset A)$

  and

- Pairs:

  $$(\mathcal{A}, U)$$

  where $\mathcal{A}$ is a subspace of $\mathcal{K}_+$ (in our case it is always finite-dimensional) and

  $$U : \mathcal{A} \to \mathcal{K}_-$$

  is an isometric operator (not necessarily surjective!)

defined as follows:

$$(\mathcal{A}, U) \mapsto \quad \text{extension} \quad A_U$$

where

$$D(A_U) = D(A) + \{\phi + U\phi : \phi \in \mathcal{A}\}$$

$$A_U(x + \phi + U\phi) = Ax + i\phi - iU\phi$$

where $x \in D(A)$, $\phi \in \mathcal{A}$.

Remark. Notice that $A_U\phi - i\phi$ is the common action of A^* on $\mathcal{K}_+$ and $A_U(U\phi) = -iU\phi$ is the common action of A^* on $\mathcal{K}_-$.

Key Observation:

Let $A_1 \supset A$ be closed symmetric extension,

$$D(A_1) = D(A) \oplus_A S$$

with S being A-closed A-symmetric subspace of $\mathcal{K}_- \oplus_A \mathcal{K}_+$. Let

$$\phi \in S, \quad \phi = \phi_- + \phi_+; \quad \phi_\pm \in \mathcal{K}_\pm\,.$$

Then

$$\boxed{||\phi_-|| = ||\phi_+||\,. \qquad (5.3)}$$ unita

It is a straightforward calculation using the rule of action of A^* on $\mathcal{K}_\pm$:
Since S is A-symmetric, one has

$$[\phi, \phi] = 0,$$

therefore,

$$0 = (A^*\phi, \phi) - (\phi, A^*\phi) = (A^*\phi_+ + A^*\phi_-, \phi_+ + \phi_-) - (\phi_+ + \phi_-, A^*\phi_+ + A^*\phi_-) =$$

$$= (i\phi_+ - i\phi_-, \phi_+ + \phi_-) - (\phi_+ + \phi_-, i\phi_+ - i\phi_-) =$$
$$= 2i(\phi_+, \phi_+) - 2i(\phi_-, \phi_-)$$

unita
and (5.3) follows.

Now we may construct the needed correspondence. We are considering only extensions with $\dim S < +\infty$ (therefore, $\dim \mathcal{A}$ is also finite): this matters only in case $n_+ = n_- = \infty$)

1)Extension $(A^* \supset) A_1 (\supset A) \implies$ **Pair** $(\mathcal{A}, U)$**.**

We have

$$D(A_1) = D(A) \oplus_A S$$
$$S < \mathcal{K}_+ \oplus_A \mathcal{K}_-$$

Let

$$\mathcal{A} := \mathrm{Proj}^{\perp_A}_{\mathcal{K}_+}(S)$$

Remark. *Clearly, this projection*

$$\mathrm{Proj}^{\perp_A}_{\mathcal{K}_+} \to \mathcal{A}$$

is one to one: if $(\phi_+, \psi^1_-) \in S$ and $(\phi_+, \psi^2_-) \in S$ then $(0, \psi^1_- - \psi^2_-) \in S$ and $||0|| = ||\psi^1_- - \psi^2_-||$ and $\psi^1_- = \psi^2_-$.

$\mathcal{A}$ is finite dimensional, and, therefore is closed in any norm.

$$U : \phi_+ \mapsto \phi_-$$

is an isometry

$$U : \mathcal{A} \to U(\mathcal{A})$$

according to (unita 5.3). Thus,

$$D(A_1) = \{x + \phi_+ + U\phi_+ \, : \, x \in D(A), \phi_+ \in \mathcal{A}\}$$

$$\boxed{A_1(x + \phi_+ + U\phi_+) = Ax + i\phi_+ - iU\phi_+}$$

(The second JvN formula)

2) Pair $(\mathcal{A}, U) \implies$ **Extension** A_1

Let $\mathcal{A} < \mathcal{K}_+$ (finite dimensional, but we may assume only $||\cdot||_H$-closedness) and let

$$U : \mathcal{A} \to \mathcal{K}_-$$

be a (not necessarily surjective) isometry.

Define the corresponding extension A_1 and its domain via the second JvN formula:

$$\boxed{\begin{gathered} D(A_1) = \{x + \phi_+ + U\phi_+ : x \in D(A), \phi_+ \in \mathcal{A}\} \\ A_1(x + \phi_+ + U\phi_+) = Ax + i\phi_+ - iU\phi_+ \end{gathered}}$$

End of Lecture 11

Lecture 12

Proposition 27. *Thus constructed $D(A_1)$ is A-closed and A-symmetric. Then, due to MAIN OBSERVATION from §5.2.1, A_1 is a closed symmetric extension of A.*

Proof. Since $\mathcal{A}$ is $||\cdot||$-closed it is $||\cdot||_A$-closed. This implies that $D(A_1)$ is A-closed.

A-symmetry:

In Proposition 26 we proved that $D(A) \oplus_A S$ is symmetric if S is A-symmetric. So suffices to show that

$$[\phi_+ + U\phi_+, \psi_+ + U\psi_+] = 0$$

if $\phi_+, \psi_+ \in \mathcal{A}$.

This is again a straightforward calculation:

$$\begin{gathered} [\phi_+ + U\phi_+, \psi_+ + U\psi_+] = \\ = (A^*(\phi_+ + U\phi_+), \psi_+ + U\psi_+) - (\phi_+ + U\phi_+, A^*(\psi_+ + U\psi_+)) = \\ (i\phi_+ - iU\phi_+, \psi_+ + U\psi_+) - (\phi_+ + U\phi_+, i\psi_+ - iU\psi_+) = \dots \\ = 2i\Big((\phi_+, \psi_+) - (U\phi_+, U\psi_+)\Big) \end{gathered} \tag{5.4}$$

zero

Due to relation $(z, z) = (Uz, Uz)$ and the polarization identity

$$(Ux, Uy) = \frac{1}{4} \sum_{\epsilon = \pm 1, \pm i} \epsilon(U(x + \epsilon y), U(x + \epsilon y)) = \frac{1}{4} \sum_{\epsilon = \pm 1, \pm i} \epsilon(x + \epsilon y, x + \epsilon y) = (x, y)$$

expression (zero 5.4) vanishes.

5.3.1 Self-adjoint extensions

Now let us compare deficiency indices of the symmetric operator A and its extension A_1. We have

$$(A_1 + iI)(x + \phi_+ + U\phi_+) = Ax + i\phi_+ - iU\phi_+ iIx + i\phi_+ iU\phi_+ =$$

$$= (A + iI)x + 2i\phi_+$$

with $(A + iI)x \in R(A + iI)$ and $2i\phi_+ \in \mathcal{K}_+ = R(A + iI)^\perp$.

Thus,

$$\operatorname{codim} R(A_1 + iI) = \operatorname{codim} R(A + iI) - \dim \mathcal{A}$$

Similarly,

$$(A_1 - iI)(x + \phi_+ + U\phi_+) = (A - iI)x - 2iU\phi_+$$

with $(A - iI)x \in R(A - iI)$ and $2iU\phi_+ \in \mathcal{K}_- = R(A - iI)^\perp$. Thus,

$$\operatorname{codim} R(A_1 - iI) = \operatorname{codim} R(A - iI) - \dim \mathcal{A}$$

Equivalently:

$$\boxed{\begin{gathered} n_+(A_1) = n_+(A) - \dim \mathcal{A} \\[1em] n_-(A_1) = n_-(A) - \dim \mathcal{A} \end{gathered}}$$

Thus,

$$\boxed{n_-(A) \neq n_+(A) \implies n_-(A_1) \neq n_+(A_1)}$$

and

$\boxed{n_-(A) \neq n_+(A) \implies \text{the operator } A \text{ does not have self-adjoint extensions}}$

and

If $n_+(A) = n_-(A)$ $(\neq \infty)$
then any isometry

$$U : \mathcal{A} = \mathcal{K}_+ \to \mathcal{K}_-$$

$(\dim \mathcal{A} = n_+(A) = n_-(A)!!!)$
defines a self-adjoint extension of A and all s. a. extensions are obtained in this way.

Once again:

For a self-adjoint extension $\tilde{A}$ of a closed symmetric operator A one has

$$D(\tilde{A}) = \{x + \phi_+ + U\phi_+ : x \in D(A), \phi_+ \in \mathcal{K}_+\}$$

$$\tilde{A}(x + \phi_+ + U\phi_+) = Ax + i\phi_+ - iU\phi_+$$

with some unitary bijection

$$U : \mathcal{K}_+ \to \mathcal{K}_-$$

5.3.2 Example

We use the results of Section 2. Consider a closed symmetric operator $A = i\frac{d}{dx}$ (we suppress "-" for convenience) in $L_2([0,1])$ with $D(A) = H_0^1[0,1]$. Then $D(A^*) = H^1[0,1]$ and $A^*u = iu'$.

Let $u \in \mathcal{K}_+ = \mathrm{Ker}\,(A^* - iI)$. Then

$$iu' - iu = 0$$

and $u = Ce^x$. Similarly,

$$\mathcal{K}_- = \mathrm{Ker}\,(A^* + iI) = \{Ce^{-x}\}$$

Thus,

$$n_-(A) = n_+(A) = 1$$

Any unitary operator

$$U : \mathcal{K}_+ = \mathrm{LinSpan}\,(e^x) \to \mathcal{K}_- = \mathrm{LinSpan}\,(e^{-x})$$

is of the form

$$f_+ \mapsto e^{i\alpha} f_-$$

with $f_\pm \in \mathcal{K}_\pm$, $||f_\pm|| = 1$. Since

$$\int_0^1 e^{2x}\,dx = \frac{e^2 - 1}{2}$$

and

$$\int_0^1 e^{-2x}\,dx = \frac{e^2 - 1}{2e^2}\,,$$

one gets

$$f_+ = \sqrt{\frac{2}{e^2 - 1}}e^x$$

and

$$f_- = e\sqrt{\frac{2}{e^2 - 1}}e^{-x}$$

Thus all the self-adjoint extensions, $A_\alpha \supset A$ are numbered by $\alpha \in [0, 2\pi)$. One has

$$D(A_\alpha) = \{\phi + C\sqrt{\frac{2}{e^2 - 1}}e^x + Ce^{i\alpha}e\sqrt{\frac{2}{e^2 - 1}}e^{-x} : \phi \in H_0^1, C \in \mathbb{C}\}$$

Now, observe that for $\psi \in D(A_\alpha)$ one has

$$\psi(0) = 0 + C\sqrt{\frac{2}{e^2-1}} + Ce^{i\alpha}e\sqrt{\frac{2}{e^2-1}}$$

$$\psi(1) = 0 + C\sqrt{\frac{2}{e^2-1}}e + Ce^{i\alpha}\sqrt{\frac{2}{e^2-1}}$$

and

$$\frac{\psi(1)}{\psi(0)} = \frac{1+e^{i\alpha}e}{e+e^{i\alpha}} =: \gamma\,.$$

Clearly,

$$|\gamma| = 1$$

which proves the claim 1) from the end of Section 2.1 (before Definition 9). Claim 2) is left as an exercise: see hints below.

Hints: 1)Solve

$$\begin{cases} u' - i\lambda u = 0 \\ u(1) = \gamma u \end{cases}$$

This gives $u = Ce^{i\lambda t}$; $\gamma = e^{-i\phi} = e^{-\lambda} \iff \lambda = \lambda_k = \phi + 2\pi k$.

2)Prove that $\lambda \neq \lambda_k \implies \exists (A_\alpha - \lambda I)^{-1}$ which is bounded and defined on $L_2[0,1]$,

$$(A_\alpha - \lambda I)^{-1} f = e^{i\lambda t}\left[\frac{1}{e^{i\phi}-e^{i\lambda}}\int_0^1 e^{i\lambda s} i f(s)\,ds + \int_0^t e^{-i\lambda s} i f(s)\,ds\right].$$

END OF WINTER 2022 COURSE

6 Pseudolaplacians in $\mathbb{R}^d$

6.1 Sobolev spaces

For positive integer k define

$$H^k(\mathbb{R}^n) = W_2^k(\mathbb{R}^n) := \{f \in L_2 : \forall\alpha : |\alpha| \le k \;\; D^\alpha f \in L_2\}$$

or

$$\{u \in S'(\mathbb{R}^n) : (1+|\xi|^2)^{k/2}\hat{u} \in L_2(\mathbb{R}^n\}\,.$$

The latter definition extends to any real s:

$$H^s(\mathbb{R}^n) := \{u \in S'(\mathbb{R}^n) : (1+|\xi|^2)^{s/2}\hat{u} \in L_2(\mathbb{R}^n\}\,.$$

We will need three standard **Sobolev's embedding theorems**

Proposition 28. *1.* $s > n/2 \implies H^s(\mathbb{R}^n) \subset C_b(\mathbb{R}^n)$ *(b means bounded)*

2. $0 < \alpha < 1 \implies H^{n/2+\alpha} \subset \mathrm{Lip}^\alpha(\mathbb{R}^n)$

($\mathrm{Lip}^\alpha(\mathbb{R}^n) \iff$ *u is bounded and* $|u(x+y) - u(y)| \le C|y|^\alpha$*)*

3. $u \in H^{n/2+1}(\mathbb{R}^n) \implies$

$$|u(x+y) - u(x)| \le C|y| \left(\log \frac{1}{|y|}\right)^{1/2}$$

(still $\to 0$ *as* $y \to 0$.*)*

Proof.

1) Simple:

$$|u(x)| = \left|\int_{\mathbb{R}^n} e^{ix\xi} \hat{u}(\xi)\, d\xi\right| \le \int_{\mathbb{R}^n} |\hat{u}(\xi)| d\xi =$$

$$\int_{\mathbb{R}^n} |\hat{u}(\xi)|(1+|\xi|^2)^{s/2}(1+|\xi|^2)^{-s/2} d\xi \le \left(\int_{\mathbb{R}^n} |\hat{u}(\xi)|^2 (1+|\xi|^2)^s d\xi\right)^{1/2} \left(\int_{\mathbb{R}^n} \frac{d\xi}{(1+|\xi|^2)^s}\right)^{1/2}$$

and the last factor is finite if $s > n/2$.

2) and 3)Similarly but longer:

$$|u(x+y) - u(x)| = \left|\int_{\mathbb{R}^n} \hat{u}(\xi) e^{ix\xi}(e^{iy\xi} - 1)\, d\xi\right| \le$$

$$\le \int_{\mathbb{R}^n} |\hat{u}(\xi)| |e^{iy\xi} - 1|\, d\xi \le$$

$$\le \left(\int_{\mathbb{R}^n} |\hat{u}(\xi)|^2 (1+|\xi|^2)^s d\xi\right)^{1/2} \times$$

$$\times \left(\int_{\mathbb{R}^n} (1+|\xi|^2)^{-s} |e^{iy\xi} - 1|^2\, d\xi\right)^{1/2}$$

with $s = n/2 + \alpha$. Thus, one has to estimate the last factor.

WLOG one can assume that $|y| \le 1/2$. Notice that $|iy\xi| \le 1$ if $|\xi| \le 1/|y|$ and for $|z| \le 1$ one has $\left|\frac{e^z - 1}{z}\right| \le C$.

Therefore

$$\int_{\mathbb{R}^n} (1+|\xi|^2)^{-n/2-\alpha} |e^{iy\xi} - 1|^2\, d\xi \le$$

,

$$\int_{|\xi| \le 1/|y|} (1+|\xi|^2)^{-n/2-\alpha} \left|\frac{e^{iy\xi} - 1}{iy\xi}\right|^2 |y|^2 |\xi|^2\, d\xi +$$

$$4 \int_{|\xi| \ge 1/|y|} (1+|\xi|^2)^{-n/2-\alpha} d\xi \le$$

$$\le C(A(y)|y|^2 + B(y))$$

with

$$A(y) = \int_{|\xi| \le 1/|y|} (1+|\xi|^2)^{-n/2-\alpha} |\xi|^2\, d\xi$$

and

$$B(y) = \int_{|\xi| \ge 1/|y|} (1+|\xi|^2)^{-n/2-\alpha} d\xi$$

Since $|y| \le 1/2$ and $1/|y| \ge 2$,

$$A(y) = (\int_0^1 + \int_1^{1/|y|})\, dr \le$$

$$\int_0^1 (1+r^2)^{-n/2-\alpha} r^2 r^{n-1} dr +$$

(integrand $\sim r^{n+1}$ near 0, therefore, this part is $\le C$)

$$+ \int_1^{1/|y|} (r^2 + r^2)^{-n/2-\alpha} r^2 r^{n-1}\, dr$$

(this part is $\le C \int_1^{1/|y|} r^{1-2\alpha}\, dr$)

$$\le C + \begin{cases} C|y|^{2\alpha-2}, & 0 < \alpha < 1 \\ C \log \frac{1}{|y|}, & \alpha = 1 \end{cases}$$

Additionally.

$$B(y) \le C \int_{1/|y|}^{\infty} (1+r^2)^{-n/2-\alpha} r^{n-1}\, dr \le$$

$$C \int_{1/|y|}^{\infty} r^{-2\alpha-1}\, dr = C|y|^{2\alpha}$$

Finally,

$$\int_{\mathbb{R}^n} (1+|\xi|^2)^{-n/2-\alpha} |e^{iy\xi} - 1|^2\, d\xi \le$$

$$\le C \left(|y|^2 + |y|^2 \begin{cases} |y|^{2\alpha-2} \\ \log \frac{1}{|y|} \end{cases} + |y|^{2\alpha} \right) \le$$

$$C \begin{cases} |y|^{2\alpha}, & 0 < \alpha < 1 \\ |y|^2 \log \frac{1}{|y|}, & \alpha = 1 \end{cases}$$

We will need also

Proposition 29. *$C_0^\infty(\mathbb{R}^n)$ is dense in $W_2^k(\mathbb{R}^n)$.*

Proof (a sketch).

1) Take smooth χ_R with support in $\{|x| \le R+1\}$ and $\chi(x) = 1$ for $|x| \le R$. Then $\chi_R u \to u$ as $R \to \infty$ for any $u \in W_2^k(\mathbb{R}^n)$. 2)Take ω_ϵ: supp$\omega_\epsilon \to \{0\}$, $\int_{\mathbb{R}^n} \omega_\epsilon = 1$. Then

$$\omega_\epsilon * (\chi_R u) \to \chi_R u$$

as $\epsilon \to 0$ in $W_2^k(\mathbb{R}^n)$. It remains to notice that

$$\omega_\epsilon * (\chi_R u) \in C_0^\infty(\mathbb{R}^n)$$

for any $u \in W_2^k(\mathbb{R}^n)$.

6.2 Pseudolaplacians

Consider the Laplace operator Δ in $L_2(\mathbb{R}^d)$ with

$$D(\Delta) := C_0^\infty(\mathbb{R}^d \setminus \{O\}) .$$

We will show that this operator is essentially self-adjoint for $d \geq 4$ and has infinitely many self-adjoint extensions for $d \leq 3$. The latter extensions are called "pseudolaplacians". They present mathematical models for Schroedinger operators with δ-function potential:

$$-\Delta + \delta .$$

6.2.1 Domain of the closure for $d \geq 4$

Proposition 30. *Let*

$$L := \Delta\Big|_{C_0^\infty(\mathbb{R}^n)}$$

and let

$$\boxed{d \geq 4}$$

Then

$$D(\overline{L}) = W_2^2(\mathbb{R}^d) = D(\mathbf{\Delta})$$

where $\mathbf{\Delta}$ *is the standard self-adjoint Laplacian in* $\mathbb{R}^d$*.*

Proof. Observe that, since the graph norm $|||u|||^2 = ||u||^2 + ||\Delta u||^2$ is equivalent to the W_2^2-norm, one has the inclusion $D(\overline{L}) \subset W_2^2$. Since $C_0^\infty(\mathbb{R}^d)$ is dense in W_2^2 it suffices to show that it is possible to approximate any function ψ from $C_0^\infty(\mathbb{R}^d)$ by functions $u_n \in C_0^\infty(\mathbb{R}^d \setminus \{O\})$ **in the W_2^2-norm** (or, what is the same, in the graph norm).

Let $\phi(x) = \tilde{\phi}(|x|)$, where $\tilde{\phi}(t)$ is smooth, vanishes for $|t| > 1$ and equals 1 for $|t| \leq 1/2$. Define

$$\phi_\epsilon(x) := 1 - \phi(\frac{x}{\epsilon})$$

(ϕ_ϵ equals zero in $\epsilon/2$-ball centred at the origin and equals one outside ϵ-ball).

We will show that $C_0^\infty(\mathbb{R}^d \setminus \{O\}) \ni \phi_\epsilon\psi \to \psi$ in W_2^2 if $d > 4$ (or, what is the same, $\phi_\epsilon\psi \to \psi$ and $\Delta(\phi_\epsilon\psi) \to \Delta\psi$ in L_2 as $\epsilon \to 0$).

For $d = 4$ a less straightforward method is needed.

1)$||\phi_\epsilon\psi - \psi|| \to 0$.

One has

$$||\phi_\epsilon\psi - \psi||^2 = \int_{\mathbb{R}^d} |(1 - \phi(\frac{x}{\epsilon}))\psi - \psi|^2 = \int_{\mathbb{R}^d} |\phi(\frac{x}{\epsilon})|^2 |\psi(x)|^2 \, dx \leq$$

$$\leq C \int_{\mathbb{R}^d} |\phi(\frac{x}{\epsilon})|^2 dx = \epsilon^d C \int_{\mathbb{R}^d} |\phi(x)|^2 \, dx \to 0$$

2)$\Delta(\phi_\epsilon\psi) \to \Delta\psi$.

One has

$$||\Delta(\phi_\epsilon\psi)-\Delta\psi|| = ||(\Delta\phi_\epsilon)\psi+\nabla\phi_\epsilon\cdot\nabla\psi+\phi_\epsilon\Delta\psi-\Delta\psi|| \le$$

$$||(\Delta\phi_\epsilon)\psi||(:=A)+||\nabla\phi_\epsilon\cdot\nabla\psi||(:=B)+||\phi_\epsilon\Delta\psi-\Delta\psi||(:=C)$$

Due to Step 1, $C\to 0$ as $\epsilon\to 0$.

$$B^2=||\nabla\psi\nabla(1-\phi(x/\epsilon))||^2\le c\int_{\mathbb{R}^d}|\nabla_x\phi(x/\epsilon)|^2\,dx = c\frac{1}{\epsilon^2}\int_{\mathbb{R}^d}|(\nabla\phi)(x/\epsilon)|^2\,dx =$$

$$c\frac{1}{\epsilon^2}\epsilon^d\int_{\mathbb{R}^d}||\nabla\phi|^2\to 0$$

(since $d-2>0$.)

Similarly

$$A^2\le c\int_{\mathbb{R}^d}|\Delta_x\phi(x/\epsilon)|^2 = c\epsilon^{d-4}\int_{\mathbb{R}^d}|\Delta\phi|^2\to 0$$

(under the condition $d-4>0$.)

For the case $d=4$ (when the last estimate does not work) a special trick is needed. The same works with improved

$$\phi_\epsilon(x):=1-\tilde{\phi}\left(\left[\frac{|x|}{\epsilon}\right]^\epsilon\right)$$

where $\tilde{\phi}$ is as above (in what follows we omit tilde).

1)

$$||\phi_\epsilon\psi-\psi||^2\le c\int_{\mathbb{R}^4}|\phi\left(\left[\frac{|x|}{\epsilon}\right]^\epsilon\right)|^2dx = c_1\int_0^\infty|\phi\left(\left[\frac{r}{\epsilon}\right]^\epsilon\right)|^2r^3\,dr =$$

$(t:=(r/\epsilon)^\epsilon;\ dr=t^{1/\epsilon-1})$

$$=c\epsilon^3\int_0^1|\phi(t)|^2t^{4/\epsilon-1}dt\le$$

$(\operatorname{supp}\phi\subset[0,1])$

$$\le c_2\epsilon^3\int_0^1|\phi(t)|^2\,dt\to 0$$

2)The most difficult part is to estimate the term A with $\Delta_x\phi\left(\left[\frac{|x|}{\epsilon}\right]^\epsilon\right)$.

$$\Delta=\left(\frac{\partial}{\partial r}\right)^2+\frac{d-1}{r}\frac{\partial}{\partial r}+\frac{1}{r^2}\delta$$

with $d=4$ and spherical part, δ, playing no role (all functions depend on radial variable only).

$$\frac{\partial}{\partial r}\phi\left(\left[\frac{r}{\epsilon}\right]^\epsilon\right)=\phi'\left(\left[\frac{r}{\epsilon}\right]^\epsilon\right)\left(\frac{r}{\epsilon}\right)^{\epsilon-1}\frac{1}{\epsilon}\epsilon$$

$$\left(\frac{\partial}{\partial r}\right)^2\phi\left(\left[\frac{r}{\epsilon}\right]^\epsilon\right)=\phi''\left(\left[\frac{r}{\epsilon}\right]^\epsilon\right)\left(\frac{r}{\epsilon}\right)^{2\epsilon-2}+\phi'\left(\left[\frac{r}{\epsilon}\right]^\epsilon\right)\left(\frac{r}{\epsilon}\right)^{\epsilon-2}\frac{\epsilon-1}{\epsilon}$$

$$\Delta_x \phi\left(\left[\frac{|x|}{\epsilon}\right]^\epsilon\right) =$$

$$\phi''\left(\left[\frac{r}{\epsilon}\right]^\epsilon\right) r^{2\epsilon-2}\epsilon^{2-2\epsilon} + (d-1)r^{\epsilon-2}\epsilon^{1-\epsilon}\phi'\left(\left[\frac{r}{\epsilon}\right]^\epsilon\right) + \phi'\left(\left[\frac{r}{\epsilon}\right]^\epsilon\right) r^{\epsilon-2}\epsilon^{1-\epsilon}(\epsilon-1) =$$

$$(d-2)r^{\epsilon-2}\epsilon^{1-\epsilon}\phi'\left(\left[\frac{r}{\epsilon}\right]^\epsilon\right) + \phi''\left(\left[\frac{r}{\epsilon}\right]^\epsilon\right) r^{2\epsilon-2}\epsilon^{2-2\epsilon} + \phi'\left(\left[\frac{r}{\epsilon}\right]^\epsilon\right) r^{\epsilon-2}\epsilon^{2-\epsilon}$$

Since ϕ', ϕ'' factors are bounded, only $|\epsilon^{1-\epsilon}r^{\epsilon-2}|^2$, $|\epsilon^{2-2\epsilon}r^{2\epsilon-2}|^2$ and $|r^{\epsilon-2}\epsilon^{2-\epsilon}|^2$ terms matter. Integration goes over $0 \le r \le \epsilon$, jacobian is $r^{d-1} = r^3$. The worst (= the biggest) term is the first. One has

$$\int_0^\epsilon \epsilon^{2-2\epsilon} r^{2\epsilon-4} r^3 \, dr = \frac{\epsilon}{2} \to 0$$

as $\epsilon \to 0$.

6.2.2 Domain of the closure for $d = 1, 2, 3$

1)$d = 3$

One has

$$W_2^2(\mathbb{R}^3) = H^2(\mathbb{R}^3) \subset \mathrm{Lip}^{\,1/2}$$
$$H^{3/2+1/2} \subset \mathrm{Lip}^{\,1/2}$$

(for $d = 4, 5, \ldots$ this is wrong!) Thus,

$$D(\overline{L}) \subset \{\psi \in W_2^2(\mathbb{R}^3) : \psi(0) = 0\}$$

Exercise: Show that one can approximate any ψ from $C_0^\infty(\mathbb{R}^3)$ such that $\psi(0) = 0$ and $|\psi(x)| \le C|x|^{1/2}$ in $B(O, \epsilon)$ in the graph norm by $u_n \in C_0^\infty(\mathbb{R}^3 \setminus \{O\})$. Act as in case $d = 4$ (take the same ψ_ϵ).

From this exercise follows the equality

$$D(\overline{L}) = \{\psi \in W_2^2(\mathbb{R}^3) : \psi(0) = 0\}$$

2)$d = 2$

$$2 = \frac{2}{2} + 1$$

Thus, the functions ψ from the domain of the closure, being in $W_2^2(\mathbb{R}^2)$ must satisfy

$$|\psi(x)| = |\psi(x) - \psi(0)| \le C|x| \left(\log \frac{1}{|x|}\right)^{1/2}$$

Exercise: Similarly to the case $d = 3$, show that

$$D(\overline{L}) = \{\psi \in W_2^2(\mathbb{R}^2) : \psi(0) = 0\}$$

3)$d = 1$ (Relevant to the theory of quantum graphs). Let

$$\psi \in D(\overline{L}) \subset W_2^2(\mathbb{R}^1)\,.$$

Then $\psi' \in H^1(\mathbb{R}^1)$. Since $1 = \frac{1}{2} + \frac{1}{2}$, $\psi' \in \mathrm{Lip}^{1/2}$.

$$D(\overline{L}) = \{\psi \in W_2^2(\mathbb{R}^1) : \psi(0) = \psi'(0) = 0\}$$

6.2.3 Self-adjoint extensions of L

1) Case $d \geq 4$.

We will show that $\mathcal{K}_- = \mathcal{K}_+ = \{0\}$ and, thus, L is essentially self-adjoint.

Let $u \in \mathcal{K}_+ = R(\overline{L} + iI)^\perp$. Then

$$(u, (\overline{L} + iI)f) = 0$$

for any $f \in D(\overline{L}) = W_2^2(\mathbb{R}^d)$. Using Plancherel theorem, one gets

$$(\hat{u}, (|\xi|^2 + i)\hat{f}) = 0$$

or

$$(\hat{u}(|\xi|^2 - i), \hat{f}) = 0$$

for any $f \in W_2^2 \supset S(\mathbb{R}^d)$.

This implies

$$\hat{u}(|\xi|^2 - i) = 0$$

and $u = 0$.

(and similarly for $\mathcal{K}_-$.)

2)Case $d = 2, 3$.

We will show that in this case $n_+ = n_- = 1$ and find all the self-adjoint extensions. Let $u \in \mathcal{K}_+$. As before, this implies

$$(\hat{u}, (|\xi|^2 + i)\hat{f}) = 0$$

for any $f \in H^2$ such that $f(0) = 0$ or, what is the same, for any $\hat{f} \in (1 + |\xi|^2)^{-1} L_2(\mathbb{R}^d)$ such that

$$0 = \int_{\mathbb{R}^d} \hat{f}(\xi)\, d\xi$$

(the last integral equals to $f(0)$ due to Fourier inversion formula).

Choose $\phi \in S(\mathbb{R}^d)$ such that

$$\int_{\mathbb{R}^d} \hat{\phi}(\xi)\, d\xi = 1\,.$$

Then for any $f \in H^2$ the function

$$g = \hat{f} - \int_{\mathbb{R}^d} \hat{f}(\xi)\, d\xi \cdot \hat{\phi}$$

belongs to $(1 + |\xi|^2)^{-1} L_2(\mathbb{R}^d)$ and satisfies $\int_{\mathbb{R}^d} g = 0$.

Thus, for any $f \in H^2 = W_2^2$

$$\left(\hat{u}, (|\xi|^2 + i)(\hat{f} - \int \hat{f} \cdot \hat{\phi})\right) = 0$$

or

$$((\xi|^2 - i)\hat{u}, \hat{f}) - (C, \hat{f}) = 0$$

with

$$C = ((|\xi|^2 - i)\hat{u}, \hat{\phi})$$

Therefore,

$$(|\xi|^2 - i)\hat{u} = C$$

and

$$u = C\mathcal{F}^{-1}\left(\frac{1}{|\xi|^2 - i}\right)$$

Thus, $\dim \mathcal{K}_+ = 1$.

Similarly $\dim \mathcal{K}_- = 1$ and

$$\mathcal{K}_- = \{C\mathcal{F}^{-1}\left(\frac{1}{|\xi|^2 + i}\right)\}$$

Now the second JvN formula gives all the self-adjoint extensions $L_\theta \supset L$ via

$$D(L_\theta) = \{\phi + \beta\mathcal{F}^{-1}\left(\frac{1}{|\xi|^2 - i}\right) + \beta e^{i\theta}\mathcal{F}^{-1}\left(\frac{1}{|\xi|^2 + i}\right) : \beta \in \mathbb{C}; \phi \in W_2^2(\mathbb{R}^d), \phi(0) = 0\}$$

with $\theta \in [0, 2\pi)$.

If $\theta = \pi$ then we get the ordinary self-adjoint Laplacian $\mathbf{\Delta}$ with domain W_2^2:

$$\phi + \beta\mathcal{F}^{-1}\left(\frac{1}{|\xi|^2 - i}\right) + \beta e^{i\pi}\mathcal{F}^{-1}\left(\frac{1}{|\xi|^2 + i}\right) = \phi + \beta\mathcal{F}^{-1}\frac{2i}{|\xi|^4 + 1} \in W_2^2\,.$$

Moreover, on the one hand (due to JvN-2)

$$L_\pi(\phi + \beta\mathcal{F}^{-1}\left(\frac{1}{|\xi|^2 - i}\right) + \beta e^{i\pi}\mathcal{F}^{-1}\left(\frac{1}{|\xi|^2 + i}\right)) = \Delta\phi + i\beta\mathcal{F}^{-1}\left(\frac{1}{|\xi|^2 - i}\right) - i(-1)\beta\mathcal{F}^{-1}\left(\frac{1}{|\xi|^2 + i}\right),$$

on the other hand

$$\mathbf{\Delta}(\phi + \beta\mathcal{F}^{-1}\left(\frac{1}{|\xi|^2 - i}\right) + \beta e^{i\pi}\mathcal{F}^{-1}\left(\frac{1}{|\xi|^2 + i}\right)) = \Delta\phi + \beta\mathcal{F}^{-1}\left(\frac{|\xi|^2}{|\xi|^2 - i} - \frac{|\xi|^2}{|\xi|^2 + i}\right)$$

and the right hand sides coincide.

7 Krein formula

Let A be closed symmetric with

$$n_-(A) = n_+(A) = n < +\infty$$

and let $A_1 \supset A$ and $A_2 \supset A$ be two self-adjoint extentions of A, $A_1^* = A_1$, $A_2^* = A_2$.

krein **Proposition 31.** *Let $\lambda \in (\mathrm{Spectrum}\,(A_1) \cup \mathrm{Spectrum},(A_2))^c$ then the difference of the resolvents*

$$R(\lambda; A_1) - R(\lambda; A_2)$$

is a finite-rank operator. It acts as follows. One has

$$H = \overline{R(A - \lambda I)} \oplus \mathrm{Ker}\,(A^* - \bar{\lambda} I)$$

$$H = \overline{R(A - \bar{\lambda} I)} \oplus \mathrm{Ker}\,(A^* - \lambda I)$$

(in case $\Im\lambda \neq 0$ one does not need the closure).

The operator $R(\lambda; A_1) - R(\lambda; A_2)$

1. *sends* $\overline{R(A - \lambda I)}$ *to zero*
2. *maps* $\mathrm{Ker}\,(A^* - \bar{\lambda} I)$ *to* $\mathrm{Ker}\,(A^* - \lambda I)$.

Reminder: $H = \overline{T} \oplus \mathrm{Ker}\, T^*$; $R(A - \lambda)$ is closed if $\Im \lambda \neq 0$; $\dim \mathrm{Ker}\,(A^* - \lambda I) = \dim \mathrm{Ker}\,(A^* - \bar{\lambda} I) = n$.

Proof. 1) Let $f \in R(A - \lambda I)$ then $f = (A - \lambda I)x$ with $x \in D(A)$. Since $A \subset A_1$ and $A \subset A_2$, one has

$$[(A_1 - \lambda I)^{-1} - (A_2 - \lambda I)^{-1}]f = x - x = 0$$

If λ is real then one has to separately consider the case $f \in \overline{R(A - \lambda I)} \setminus R(A - \lambda I)$. Then

$$[(A_1 - \lambda I)^{-1} - (A_2 - \lambda I)^{-1}]f = 0$$

because both the resolvents are bounded operators in H.

2)Let $f \in \mathrm{Ker}\,(A^* - \bar{\lambda} I)$ We are to prove that

$$[(A_1 - \lambda I)^{-1} - (A_2 - \lambda I)^{-1}]f \in \mathrm{Ker}\,(A^* - \lambda I)\,.$$

Due to equality

$$\overline{R(A - \bar{\lambda} I)} \oplus \mathrm{Ker}\,(A^* - \lambda I) = H$$

it suffices to prove that for any $h \in R(A - \bar{\lambda} I)$ one has

$$([(A_1 - \lambda I)^{-1} - (A_2 - \lambda I)^{-1}]f, h) = 0\,.$$

This is simple:

$$([(A_1 - \lambda I)^{-1} - (A_2 - \lambda I)^{-1}]f, h) = (f, (A_1 - \bar{\lambda} I)^{-1} - (A_2 - \bar{\lambda} I)^{-1}]h) = (f, 0) = 0\,.$$

The proposition is proved.

Now, **for a fixed** $\lambda \in (\mathrm{Spectrum}\,(A_1) \cup \mathrm{Spectrum},(A_2))^c$**,** choose

$$g_1(\lambda), \ldots, g_n(\lambda)$$

- a basis of $\mathrm{Ker}\,(A^* - \lambda)$

and

$$g_1(\bar{\lambda}), \ldots, g_n(\bar{\lambda})$$

- a basis of $\mathrm{Ker}\,(A^* - \bar{\lambda})$.

Proposition krein 31 implies

$$\forall f \in H:$$

$$[(A_1 - \lambda I)^{-1} - (A_2 - \lambda I)^{-1}]f = \sum_{k=1}^{n} c_k(f) g_k(\lambda)$$

where $c_k(\cdot)$ are bounded linear functionals on h. From Riesz theorem one gets

$$c_k(f) = (f, h_k)$$

for some $h_k \in H$.

Moreover, one has

$$(f, h_k) = 0$$

for $f \in R(A - \lambda I)$, therefore,

$$h_k \in \mathrm{Ker}\,(A^* - \bar{\lambda} I)$$

or

$$h_k = \sum_{l=1}^{n} \bar{p}_{kl}(\lambda) g_l(\bar{\lambda})$$

which implies

$$[(A_1 - \lambda I)^{-1} - (A_2 - \lambda I)^{-1}]f = \sum_{k,l=1}^{n} p_{kl}(\lambda)(f, g_l(\bar{\lambda})) g_k(\lambda).$$

This proves *M. G. Krein's formula* for the difference of the resolvents of two self-adjoint extensions, A_1 and A_2, of a given closed symmetric operator A:

$$\boxed{R(\lambda; A_1) - R(\lambda; A_2) = \sum_{k,l=1}^{n} p_{kl}(\lambda)(\,\cdot\,, g_l(\bar{\lambda})) g_k(\lambda)\,.}$$

8 Appendix 1: Spectral theorem for compact self-adjoint operators

Initially, this was supposed to be an introduction to the course.

To warm up we recall the most standard (contained in almost all basic texts in FA) proof of the ST for compact s.-a. operators.

$\mathcal{H}$ - separable (complex) Hilbert space; A - compact symmetric (=s. a. for bounded case) operator:

$$\forall x, y \in \mathcal{H}\ (Ax, y) = (x, Ay);$$

$$A(b\mathit{ounded}) = \text{precompact}$$

λ - eigenvalue of A; V_λ - corresponding eigenspace (the set of all eigenvectors corresponding to λ) (different eigenspaces are mutually orthogonal! - **easy exercise 1**). Λ - the set of all eigenvalues of A (all are real! - **easy exercise 2**).

Spectral Theorem:

$$\mathcal{H} = \bigoplus_{\lambda \in \Lambda} V_\lambda$$

(direct orthogonal sum).

Addendum:

- if $\Lambda \ni \lambda \neq 0$ then V_λ is finite dimensional
- For any $\epsilon > 0$ $\Lambda \cap \{|x| > \epsilon\}$ is a finite set.

  Comment: Ker A=V_0 may be infinite dimensional (as well as finite dimensional or even zero). If $\mathcal{H}$ is infinite dimensional then $\lambda = 0$ is the (only) limit point of Λ.

  One can organize the non-zero eigenvalues in the sequence $\lambda_1, \lambda_2, \ldots,$ such that $|\lambda_n| \geq |\lambda_{n+1}|$ (each eigenvalue is repeated according to multiplicity).
- If $\{e_n\}$ is the orthonormal system of eigenvectors corresponding to non-zero eigenvalues λ_n (in each eigenspace an orthonormal basis is chosen: take the union of all these bases) then $\forall x \in \mathcal{H}$

$$Ax = \sum_n \lambda_n (x, e_n) e_n$$

  (convergence in $\mathcal{H}$).

The proof models finite dimensional linear algebra: why hermitian matrices are diagonalizable? Trivial:

A) $\det(A - \lambda I) = 0$ has a solution λ_1 in $\mathbb{C}$ (Main theorem of algebra). (In fact, real.) So there exists an eigenvector v_1. Since A is hermitian, the orthogonal complement $H_1 = v_1^\perp$ is A-invariant (**easy exercise 3**).

B) $A_1 := A\Big|_{v_1^\perp}$. A_1 - hermitian (**easy exercise 4**). Continue.

For infinite dimensional case: A) (existence of an eigenvector) is non trivial; B) is essentially the same as in the finite dimensional case.

Proof.

Basic ingredients:

1)Polarization identity (for any linear operator A).

$$(Ax, y) = \frac{1}{4} \sum_{\epsilon = \pm 1, \pm i} \epsilon \left(A(x + \epsilon y), (x + \epsilon y)\right)$$

(Remark: mind that for us $(\cdot, \cdot)$ is anti-linear w. r. t. **the second** argument. Many authors (more close to physics) use another agreement (bra -(c)ket). In that case the r. h. s. should be changed to conjugate.)

(**Easy excercise 5**)

2)Parallelogram identity

$$||x - y||^2 + ||x + y||^2 = 2||x||^2 + 2||y||^2$$

(**Make a picture and prove - trivial exercise 6**).

3) Important property: for any hermitian (= bounded self-adjoint) A

$$\boxed{||A|| = \sup_{||x||=1}|(Ax,x)| \qquad (8.1)}$$

(**Exercise**: $\forall x \;\; (Ax,x) \in \mathbb{R}$ iff $A = A^*$. Hint: Use polarization identity for the hard part.)

Proof of (8.1). $N_A := \sup_{||x||=1}(Ax,x)$. Clearly, $N_A \le ||A||$ (follows from the Cauchy inequality.)

Let us show that $||A|| \le N_A$.

Clearly,

$$||A|| = \sup_{||x||=||y||=1}|(Ax,y)|\,. \tag{8.2}$$

In fact, $\sup_{||x||=||y||=1}|(Ax,y)| \ge \sup_{||x||=1}(Ax, Ax/||Ax||) = \sup_{||x||=1}||Ax|| = ||A||$. The opposite inequality again follows from Cauchy.

Moreover,

$$||A|| = \sup_{||x||=||y||=1}|\Re(Ax,y)| \tag{8.3}$$

Using (8.2) choose x_0, y_0, $||x_0|| = ||y_0|| = 1$ such that

$$|(Ax_0,y_0)| \ge ||A|| - \epsilon$$

One has

$$(Ax_0,y_0) = |(Ax_0,y_0)|e^{i\alpha}$$

and

$$|\Re(A(e^{-i\alpha}x_0),y_0)| = |(A(e^{-i\alpha}x_0),y_0)| = |(Ax_0,y_0)| \ge ||A|| - \epsilon$$

Thus,

$$\sup_{||z||=||y||=1}|\Re(Az,y)| \ge ||A||.$$

The opposite inequality is trivial.

Now from polarization identity

$$|\Re(Au,v)| = \left|\frac{1}{4}(A(u+v),u+v) - \frac{1}{4}(A(u-v),u-v)\right| \le$$

$$\frac{1}{4}(|(A(u+v),u+v)| + |(A(u-v),u-v)|) \le$$

$$\frac{1}{4}(N_A||u+v||^2 + N_A||u-v||^2) =$$

$$\frac{1}{4}N_A(2||u||^2 + 2||v||^2)$$

(parallelogram identity was used at the last step). Thus,

$$||A|| = \sup_{||u||=||v||=1}|\Re(Au,v)| \le N_A\,.$$

Remark. From (8.1) one gets the equality

$$||A^*A|| = \sup_{||x||=1}|(A^*Ax,x)| = \sup_{||x||=1}(Ax,Ax) = ||A||^2$$

which is important in the theory of C^*-algebras.

Now we can show that a non-zero compact s. a. operator A has an eigenvector corresponding to the (nonzero) eigenvalue λ_1 with $|\lambda_1| = ||A|| \neq 0$.

Using ($\overset{\texttt{norm}}{8.1}$), choose x_n, $||x_n|| = 1$ such that

$$(Ax_n, x_n) \to \lambda_1$$

with $|\lambda_1| = ||A||$

Informally: $|(Ax_n, x_n)|$ goes to maximum, therefore, Ax_n must be almost proportional to x_n and the coefficient of proportionality should be λ_1.

That is true:

$$0 \leq ||Ax_n - \lambda_1 x_n||^2 = (Ax_n - \lambda_1 x_n, Ax_n - \lambda_1 x_n) = ||Ax_n||^2 - 2\lambda_1(Ax_n, x_n) + \lambda_1^2 =$$

$$||Ax_n||^2 - \lambda_1^2 + o(1) \leq ||A||^2 - \lambda_1^2 + o(1) = o(1)$$

Thus, $Ax_n - \lambda_1 x_n \to 0$.

Now we will use compactness of A. Since $||x_n|| = 1$, WLOG $Ax_n \to v_1$ and, therefore, $\lambda_1 x_n \to v_1$ and

$$x_n \to v_1/\lambda_1 \,.$$

Passing to $n \to \infty$, we get

$$Av_1/\lambda_1 - v_1 = 0$$

or

$$Av_1 = \lambda_1 v_1.$$

(Mind that $\lambda_1 \neq 0$ and $||x_n|| = 1$, therefore,$v_1 \neq 0$.)

The first eigenvector of A is constructed. The rest is more or less simple.

Exercise 7: Let V be the closure of the linear span of all the eigenvectors constructed one by one as in finite dimensional case. Prove that $A|_{V^\perp} = 0$.

Exercise 8: Derive all the properties formulated in the Addendum. Hint: Let there exist infinite number of (mutually orthogonal) unit eigenvectors e_n of A with eigenvalues λ_n with absolute values greater than ϵ. WLOG (compactness) Ae_n converges. But

$$||Ae_n - Ae_m||^2 = ||\lambda_n e_n - \lambda_m e_m||^2 = |\lambda_m|^2 + |\lambda_n|^2 \geq 2\epsilon^2$$

which contradicts convergence.

8.1 The same via resolvent and analyticity

Let $\lambda \neq 0$; $\lambda \in \mathbb{R}$. A - s. a., compact.

Then

1)Ker $(A + \lambda I)$ is finite-dimensional.

(Let $\{x_n\}$ be infinite orthonormal sequence from Ker $(A + \lambda I)$.

$$||Ax_m - Ax_n|| = |\lambda| \, ||x_n - x_m|| = \sqrt{2}|\lambda|$$

So $\{Ax_n\}$ contains no converging subsequence.)

2)

$$(A+\lambda I):(\mathrm{Ker}\,(A+\lambda I))^{\perp}\to(\mathrm{Ker}\,(A+\lambda I))^{\perp}$$

is a (bounded) bijection.

1)$x\in(\mathrm{Ker}\,(A+\lambda I))^{\perp}\Rightarrow(A+\lambda I)x\in(\mathrm{Ker}\,(A+\lambda I))^{\perp}$

Let $y\in\mathrm{Ker}\,(A+\lambda I)$.

$$((A+\lambda I)x,y)=(x,(A+\lambda I)y))=(x,0)=0$$

(in fact, for any x.

Injectivity: $L\cap L^{\perp}=\{0\}$.

2)(**Exercise**:

$$\mathcal{H}=\overline{\mathrm{Im}\,(B)}\oplus\mathrm{Ker}\,B^{*} \tag{8.4}$$ `range`

for any bounded (or even densely defined - the notion of the adjoint for unbounded operators will be introduced later) B)

Comment: in the future Im = I*mage* = $\mathcal{R}$ = R*ange*.

Im$(A+\lambda I)$ - closed.

Let x_n\{Ker\} (A + λI)$^{\perp}$, $(A+\lambda I)x_n\to y$. One can assume that

$$||x_n||\le C$$

. Indeed, let $x_n\to infty$. $\omega_n:=x_n/||x_n||$

$$(A+\lambda)\omega_n-y/||x_n||\to 0$$

and, therefore,

$$(A+\lambda)\omega_n\to 0$$

WLOG (compactness) $A\omega_n\to\Omega_0$. Thus, $\omega_n\to\Omega_0/\lambda$ and Thus,

$$(A+\lambda)\omega_0=0$$

But $||\omega_0||=1$ and

$$\omega_o\in\mathrm{Ker}\,(A+\lambda I)^{\perp},$$

which gives contradiction.

Now WLOG $Ax_n\to z$ and $z+\lambda x_n-y\to 0$. Thus, $x_n\to(y-z)/\lambda=x_0$ and

$$(A+\lambda)x_0=y\,.$$

Comment: A - compact; $I+A$ - Fredholm. The proof is essentially the same.

Definition 14. $R(z)=(A-zI)^{-1}$ - *resolvent*

resolvent **Theorem 4.** *Let A be self-adjoint. Then*

1) $R(z)$ is defined for $\Im z \neq 0$.

2)

$$||R(z)|| \leq 1/|\Im z|$$

3)$R(z)^* = R(\bar{z})$;

$$R(z) - R(w) = R(z)R(w)(z-w) = R(w)R(z)(z-w) \tag{8.5}$$ resid

Will be proved for unbounded operators later. The proof for bounded operators is the same - find it now as an **exercise**.

Consider the resolvent of a compact self-adjoint operator A. Let $\lambda \in \mathbb{C}$; $\lambda \neq 0$. Let $E : \mathcal{H} \to \mathrm{Ker}\,(A - \lambda I)$ be the orthogonal projection (always finite dimensional).

KEY FACT:

$$R(z) = \frac{E}{\lambda - z} + R_1(z) \tag{8.6}$$ compres

in a vicinity of λ with analytic $R_1(z)$.

Proof. $A - zI$ respects $\mathrm{Ker}(A - \lambda I) \oplus (\mathrm{Ker}(A - \lambda I))^\perp = N \oplus F$ (**easy exercise**). For $u \in N$:

$$(A - zI)u = (A - \lambda + (\lambda - z))u = (\lambda - z)u$$

So on N one has

$$(A - zI)^{-1} = \frac{1}{\lambda - z} I\,.$$

On F: Neumann series. $A - \lambda I$ - invertible, bounded (on F - see above).

$$A - zI = A - \lambda I - (z - \lambda)I = P - S$$

$$(P - S)^{-1} = ((I - SP^{-1})P)^{-1} = P^{-1}(I + \sum_{j>0}(SP^{-1})^j)$$

$$(A - zI)^{-1} = (A - \lambda I)^{-1}(I + \sum_{j>0}(z - \lambda)^j (A - \lambda I)^{-j}) =$$

$$\sum_{j \geq 0}(A - \lambda I)^{-j-1}(z - \lambda)^j\,.$$

Therefore, on $\mathcal{H}$ one has (compres 8.6).

Thus, non-zero eigenvalues of A are *isolated* (and, in particular, there are at most countable number $\{\lambda_k\}$ of them).

Let E_k be corresponding orthogonal projections (on $\mathrm{Ker}\,(A - \lambda_k I)$). Then $E_k E_j = 0$ if $k \neq j$ and $E_k^2 = E_k$.

(**Exercise**: straightforward and elementary (eigenvectors corresponding to different eigenvalues of a s. a. operator are orthogonal); or use resolvent identity (resid 8.5) and (compres 8.6) - more elegant.)

Non-zero eigenvalues: $\lambda_1, \ldots$; $\lambda_0 := 0$ For any $u \in \mathcal{H}$ the sum

$$Eu = \sum_{j \geq 1} E_j u$$

is well-defined (Bessel inequality: $||\sum ||E_j u||^2 \leq ||u||^2$); $E^2 = E$; $E^* = E$; E- projector.

$$E_0 := I - E$$

Claim.

For any $u \in \mathcal{H}$ and for any $z \in \mathbb{C} \setminus \mathbb{R}$

$$R(z)u = (A - zI)^{-1}u = \sum_{j=0}^{\infty} (\lambda_j - z)^{-1} E_j u \tag{8.7}$$ maincompres

ATTENTION: the following proof illustrates the crucial idea of this course: complex analysis applied to operator theory. Analytical properties of the resolvent are of primary importance.

Remark: the r. h. s. is well-defined:

$$|\lambda_j - z| \geq |\Im z|$$

$$||r.h.s.|| \leq ||u||/\Im z\,.$$

Consider

$$F(z) = (R(z)u, v) - \sum_0^{\infty} ((\lambda_j - z)^{-1} E_j u, v)$$

One has

1) $F(z)$ is analytic in $\mathbb{C} \setminus \{0\}$.

(explain! - easy)

2) $|F(z)| \leq 2||u||\,||v||/\Im z$

(explain! - easy)

3)$F(z) = O(|z|^{-2})$ as $z \to \infty$.

(Explain! Hints:

a)Neumann: $-\frac{1}{z-A} = -\frac{1}{I-A/z}\frac{1}{z} = -\frac{1}{z}(I + A/z + \dots)$

b)$\sum_0^{\infty}(E_j u, v) = (u, v)$ by definition of E_0.)

Lemma 14. *1, 2, and 3 imply $F = 0$ identically.*

Proof. All the Laurent coefficients

$$\int_{|z|=R} F(z) z^k\, dz$$

are equal to 0.

1)$k \leq 0$ - send $R \to \infty$.

2)Positive k. Induction $k - 2 \to k$. Genial (from Hoermander) trick:

$$\int_{|z|=R} F(z) z^k\, dz = (induction) = \int_{|z|=R} F(z) z^{k-2}(z^2 - R^2) dz =$$

$$(z = R\zeta)$$

$$\int_{|\zeta|=1} F(R\zeta) R^{k-2} \zeta^{k-2} R^2 (\zeta^2 - 1) R\, d\zeta$$

$$|F(R\zeta)| \leq \frac{2||u||\,||v||}{\Im(R\zeta)}$$

and

$$\left|\int_{|z|=R} F(z)z^k\,dz\right| \leq R^k 2||u||\,||v|| \int_{|\zeta|=1} |\zeta^2 - 1|(\Im\zeta)^{-1}|d\zeta \to 0$$

as $R \to 0$.

Recall that we have defined

$$E_0 = I - \sum_{j=1}^{\infty} E_j\,.$$

Now we are able to show that $AE_0 = 0$ i. e. that E_0 consists of eigenvectors corresponding to $\lambda_0 = 0$.

maincompres

From (8.7)

$$u = (A - zI)^{-1} \sum_0^{\infty} (\lambda_j - z)E_j u =$$

$$(A - zI)\frac{E_0 u}{(-z)} + \sum_1^{\infty} E_j u =$$

$$\frac{1}{(-z)} AE_0 u + E_0 u + \sum_1^{\infty} E_j u =$$

$$-\frac{1}{z} AE_0 u + u$$

Thus, $AE_0 u = 0$ for any u and

$$\mathcal{H} = \bigoplus_{j=0}^{\infty} \mathcal{V}_j\,,$$

with $A|_{\mathcal{V}_j} = \lambda_j I$.

9 Appendix 2: Zorn Lemma

This is written to give a technical background for the extension of the Spectral Theorem to the case of non-separable Hilbert spaces.

9.1 Weak Zorn: formulation

First Weak ZORN (WZ):

A - a partially ordered set (=poset).

Z - a chain (= a totally ordered subset) in A.

$m_0 = \sup_A Z$ or $\sup Z$ - least upper bound (in A) that is

0) $m_0 \in A$

1)m_0 is an upper bound : $\forall z \in Z\ z \leq m_0$

2)If m_1 is another upper bound then $m_0 \leq m_1$.

WZ: A is a poset such that any chain $Z \subset A$ has the least upper bound. Then A has a maximal element (i. e. an element a_0 such that $a \geq a_0$ implies $a = a_0$).

9.2 Reduction of WZ to Bourbaki lemma

Let WZ is wrong then any element a of A is not maximal, i. e. the set $U_a = \{x \in A : x > a\}$ is not empty.

Axiom of choice: one can choose an element in each U_a.

Thus, there is a function $f : A \to A$ such that $f(a) > a$ for all $a \in A$.

Existence of such a function contradicts to the following Lemma.

Bourbaki lemma: Let a poset A satisfy conditions of WZ. Consider $f : A \to A$ such that $f(a) \geq a$ for all $a \in A$. Then there exists a_0 such that

$$f(a_0) = a_0 \,.$$

9.3 Proof of Bourbaki Lemma

Idea: suppose for a while that A satisfies the conditions of WZ and is totally ordered itself. Then A has the least upper bound a_0. Thus,

1) $f(a_0) \geq a_0$ (property of f)

2)$f(a_0) \leq a_0$ (property of an upper bound)

and, therefore, $f(a_0) = a_0$.

So, one has to find a non-empty subset of A with all the properties of A (i. e. satisfying WZ conditions and conditions for f) and which is *totally ordered*.

First, formalize "a non-empty subset of A with all the properties of A (i. e. satisfying WZ conditions and conditions for f)".

Choose an element $a \in A$. A subset B of A is admissible if

1)$a \in B$

2)Any chain $Z \subset B$ has $\sup_A Z \in B$.

3)$f(B) \subset B$.

In particular, A itself is admissible.

Take the intersection M of all admissible subsets of A.

It is non-empty (contains a!) admissible (trivial) and **totally ordered** (the hardest and the central part of the proof, all the rest is trivial) and this immediately gives Bourbaki and, therefore, WZ.

9.3.1 M is totally ordered

Let us throw off all the $c \in A$ such that $c < a$ and all the c that are not comparable with a.

$$A := A \cap \{b \in A : b \geq A\}$$

All remains the same. If the statement is proved for new A then it is proved for old A. But now we will be able to find at least one extreme point (the a itself) in M in the sense of the definition below.

Definition 1: $c \in M$ is an extreme point of M if

$$x \in M; x < c \Rightarrow f(x) \leq c$$

Remark. a is an extreme point of M. (There are no x: $x < a$.)

Plan: we are going to prove that all the points of M are extreme and there are no points of M strictly between c and $f(c)$ if c is extreme. This will imply that M is totally ordered.

Proposition 1. c — extreme, $M_c = \{\text{before}\, c \quad \text{or} \quad \text{after}\, f(c)\} =:$

$$M_c := \{x \in M : x \le c \ \text{ or } \ x \ge f(c)\}$$

Then $M_c = M$ (or there is nothing strictly between c and $f(c)$ in M)

We will prove that M_c is non-empty admissible. This implies $M_c = M$ (since M is the intersection of all admissible sets).

Non-empty: $a \in M_c$! (a is comparable with all others and is smaller or equal any other element)

$f(M_c) \subset M_c$: trivial (let $m \in M_c$. Three cases $m < c$, $m = c$, $m \ge f(c)$ – in all three $f(m) \in M_c$. (Property of f ($f(x) \ge x$).

Let T be totally ordered subset of M_c. Let b be its least upper bound in M

Case 1: T completely from the left of c. Then c is an upper bound and the least upper bound b is less or equal than c, so b in M_c.

Case 2: There are elements of T from the right of $f(c)$. Then $f(c) \le b$ and again $b \in M_c$.

Proposition 2. All the elements of M are extreme.

The same trick. Let E be the set of all extreme points of M. Then E is non-empty (contains a!) and again admissible (therefore, coincides with M which is the smallest admissible). Why admissible?

1) Let $e \in E$. Why $f(e)$ again extreme? Let $x < f(e)$ (x from M, of course). We must show that $f(x) \le f(e)$. But $M_e = M$ (see Prop. 1) So either $x < e$ or $x = e$ or $x \ge f(e)$. Last case is impossible. For other two cases we have $f(x) \le f(e)$ from Property of f: If $x < e$ then $f(x) \le e$ (e- extreme) and $e \le f(e)$ (Property of f). So $f(x) \le f(e)$.

If $x = e$ then $f(x) = f(e) \le f(e)$.

So $f(E) \subset E$.

2)$T \subset E$ – totally ordered. Then $\sup_M T \in E$. Why?

Let $b = \sup_M T$.

Suppose $x \in M$ and $x < b$. (We have to prove $f(x) \le b$ and, therefore, b - extreme)

Then one can find $t \in T$ (and, therefore, extreme) such that

$$x \le t\,.$$

Why?

$$M = M_t = \{m : m \le t \text{ or } \ m \ge f(t)\}$$

If for any $t \in T$ t is not $\ge x$ then for all t from T $f(t)$ is $\le x$. Then x is an upper bound for T and, therefore, $x \ge b$ (contradiction).

Cases:

1)$x < t$. Then $f(x) \le t$ (since t – extreme). And $f(x) \le t \le b$.

2)$x = t$. Then $x < b$ But $M_x = M_t = M$. Thus, $b \ge f(x)$.

9.3.2 M – totally ordered

Let $x, y \in M$. Then x is extreme point of M (all the points of M are extreme). Thus, $M_x = M$ and either $y \le x$ or $y \ge f(x)$. But $f(x) \ge x$. So, in the second case $y \ge x$.

WZ is proved.

9.4 Strong Zorn

We do not require existence of the least upper bound. Simple upper bound suffices.

Zorn: A is a poset such that any chain $Z \subset A$ has an upper bound. Then A has a maximal element (i. e. an element a_0 such that $a \ge a_0$ implies $a = a_0$).

Very simple.

X the set of non-empty totally ordered subsets of A.

X is a poset ("less or equal"$= \subset$)

Any chain in X has the least upper bound (union of all the elements of chain).

WZ implies there exists maximal element M in X – i. e. a totally ordered subset of A which is not contained in a strictly bigger totally ordered subset of A.

Let m_0 be the upper bound of M. Then m_0 is a maximal element in A.

Suppose it is not maximal. Then there exists $z \in A$ such that $z > m_0$.

Then $M \cup \{z\}$ is totally ordered and strictly bigger than M. Contradiction.

9.5 Applications

9.5.1 Hamel basis

V - vector space over k; $\mathcal{H} = \{v_\alpha\}_{\alpha \in A}$ - Hamel basis if

1) Any vector from V is a **finite** linear combination with coefficients from k of elements from $\mathcal{H}$.

2) Any finite subset of $\mathcal{H}$ is linear independent over k.

Always exists.

Consider set X of subsets of V with property 2). Partial order: inclusion. Every chain has upper bound (Union). Zorn implies existence of the maximal element M. That is Hamel basis (if a vector from V is not a finite linear combination of the elements of M then it can be added to M producing an element of X strictly bigger than M).

Examples: 1) $\mathbb{R}$ over $\mathbb{Q}$.

$$f(x+y) = f(x)f(y)$$

Easy: $\{v_\alpha\}_{\alpha \in A}$ Hamel basis of $\mathbb{R}$ over Q. Let $f(v_\alpha)$ be arbitrary real positive numbers. Then one can define $f(x) = \prod_\beta f(v_\beta)^{q_\beta}$ where $x = \sum_\beta q_\beta v_\beta$; β runs over finite subset of A; $q_\beta \in \mathbb{Q}$.

But if f is continuous then from $f(mx_0/n) = f(x_0)^{m/n}$ and continuity one gets $f(x) = a^x$ for some a.

2) Hahn-Banach

3) Algebraic closure of a field.

10 Some basic facts from the course FA-1

10.1 Uniform boundedness principle = UBP

UBP:

$$A_\alpha : X \to Y$$

X, Y - Banach spaces, A_α - a family of linear bounded operators.

$$\forall x \in X \quad ||A_\alpha x|| \leq C(x) \Rightarrow ||A_\alpha|| \leq \text{const} < +\infty$$

APPLICATIONS (for separable Hilbert spaces)

1. **If a sequence weakly converges then it is bounded.**

   H - Hilbert space, $x_n \rightharpoonup x_0 \quad \Rightarrow \quad ||x_n|| \leq C$

   Proof. $\forall x \in H$ the sequence $< x, x_n >$ converges and, therefore, is bounded: $| < x, x_n > \leq C(x)$. Thus, UBP implies $||x_n|| \leq C$.

2. **Hilbert space is weakly complete.**

   Let $\{x_n\}_{n=1}^\infty$ be a sequence from H. Let $\forall x \in H$ the sequence $< x, x_n >$ converges (i. e. $< x, x_n - x_m > \to 0$ as $n, m \to \infty$). Then $\exists x_0 \in H$: $x \rightharpoonup x_0$.

   Proof. $l(x) := \lim < x, x_n >$. From convergence we get $| < x, x_n > | \leq C(x)$ and UBP implies $||x_n|| \leq C$. Thus $|l(x)| = |\lim < x, x_n > | \leq C||x||$ and $l(\cdot)$ is a bounded linear functional, thus (Riesz!) $l(x) =< x, x_0 >$ for some $x_0 \in H$.

3. **Balls in H are weakly compact.**

   If $||x_n|| \leq C$ then $\exists$ subsequence x_{n_k} weakly converging to some x_0 with $||x_0|| \leq C$.

   Proof. $V :=$ LinSpan$\{x_n\}_{n=1}^\infty$ (=finite linear combinations). $H = \bar{V} \bigoplus V^\perp$ Step 1. $\exists x_{n_k}$ such that $\forall x \in V < x_n, x >$ converges. Trivial: for choose a subsequence such that $< x_1, x_{n_k} >$ converges, from this subsequence a subsubsequence such that $< x_2, x_{n_{k_l}} >$ converges and so on. Then take diagonal.

   Step 2. This subsequence should converge for any x from $\bar{V}$:

   Let $f \in \bar{V}$ and let $h \in V$ and $||h - f|| \leq \epsilon$.

   $$|| < x_{n_k} - x_{n_l}, f > || \leq || < x_{n_k} - x_{n_l}, f - h > || + || < x_{n_k} - x_{n_l}, h > || \leq 2C\epsilon + \epsilon$$

   for big k and l.

   This subsequence converges for any $x \in H = \bar{V} \bigoplus V^\perp$ because $< x_{n_k}, x >= 0$ for $x \in V^\perp$.

   Thus (item 2) $x_{n_k} \rightharpoonup x_0$. It remains to show that $||x_0|| \leq C$.

   Lemma.

   $$x_n \rightharpoonup x_0 \quad \Rightarrow \liminf ||x_n|| \geq ||x_0||$$

   Proof.

   $$0 \geq < x_n - x_0, x_n - x_0 >= ||x_n||^2 - 2\Re < x_n, x_0 > + ||x_0||^2$$

the second term tends to $-2||x_0||^2$. Pass to $\liminf$.

The estimate follows from the lemma.

4. **Landau theorem.** (In the simplest form: one can also prove $l^p - l^q$ and even $L_p - L_q$ versions)

   Let $\{a_k\}_{k=1}^{\infty}$ be a number sequence and let $\forall \{x_k\} \in l^2$ the series $\sum_k a_k x_k$ converges. Then $\{a_k\} \in l^2$.

   Proof. Introduce the linear bounded functionals in l^2:

$$L_N(x) = \sum_{k=1}^{N} a_k x_k$$

   Clearly,

$$||L_N|| = \left(\sum_{k=1}^{N} |a_k|^2 \right)^{1/2}$$

   Since $\sum a_k x_k$ converges $|L_N(x)| \le C(x)$. Thus, UBP implies

$$||L_N|| \le C$$

   and, therefore, $\sum_{k=1}^{\infty} |a_k|^2 \le C$.

5. **Sesquilinear form continuous w. r. t. each argument is continuous.**

   If $\forall x, y \;\; B(\cdot, y)$ and $B(x, \cdot)$ are bounded then $|B(x,y)| \le C||x||\,||y||$.

   Proof. Step 1. $(x_n, y_n) \to 0$ implies $B(x_n, y_n) \to 0$. $L_n(\cdot) := B(x_n, \cdot)$. $\forall y \;\; |L_n(y)| \le C(y)$, Thus (UBP), $|L_n(y)| \le C||y||$ and $|B(x_n, y_n)| \le C||y_n|| \to 0$.

   Step 2. Let $\exists x_n, y_n : \;\; ||x_n|| = ||y_n|| = 1$ and $|B(x_n, y_n)| \ge n^2$.

   Then $\tilde{x}_n = \frac{x_n}{n||x_n||}$ and $\tilde{y}_n = \frac{y_n}{n||y_n||}$ tend to zero and

$$|B(\tilde{x}_n, \tilde{y}_n)| \ge 1 \quad ?!$$

   Thus, $|B(x,y)| \le C||x||\,||y||$.

10.2 Compact operators in Hilbert spaces: basic properties

Definition: Bounded to precompact (= with compact closure).

1. **Uniform approximation by finite-dimensional operators.**

   A - compact, $\forall \epsilon > 0$ there exists finite-dimensional B: $||A - B|| \le \epsilon$.

   Proof.

   Remark. Works only if the target space is Hilbert (there are orthogonal projectors on the subspaces!!) Enflo's counterexample for Banach space as a target.

Let $y_1, \ldots, y_N$ be ϵ-net for $A(B(1))$. Then $\forall x \in B(1)$

$$||Ax - y_k|| \leq \epsilon$$

for some k.

$$V := \text{LinSpan}(\{y_k\})$$

Let P_V be orthogonal projection $P_V : H \to V$.

$$B := P_V A$$

One has

$$||P_V A - A|| = \sup_{||x||=1} ||P_V Ax - Ax|| \leq 2\epsilon$$

since

$$||P_v Ax - Ax|| = ||P_V Ax - y_k + y_k - Ax|| \leq ||P_v(Ax - y_k)|| + ||Ax - y_k||$$

and $||P_V|| = 1$.

2. **Weakly converging to converging.**

   A- compact, $x_n \rightharpoonup x_0$. Then $Ax_n \to Ax_0$.

   Proof. B - finite-dimensional ϵ-approximation.

   $$||A(x_k - x_0)|| \leq ||(A - B)(x_k - x_0)|| + ||Bx_k - Bx_0||$$

   For the first term: $||x_k|| \leq C$ (see above) and $||(A - B)(x_k - x_0)|| \leq C\epsilon$ For the second term:

   $$B = \sum_{k=1}^{N} < \cdot, g_k > f_k$$

   and $< x_k - x_0, g_k > \to 0$.

3. **Other standard properties:** A_n compact, $||A_n - A|| \to 0$ then A - compact;

   A - compact then A^* - compact

   A - compact B - bounded, then AB and BA - compact, etc

Geometric Lemma

$\dim H = \infty \Rightarrow I$ **is not compact**

Let $x_1, \ldots, x_n, \ldots$ be linearly independent and $L_n := \text{LinSpan}\,\{x_1, \ldots, x_n\}$. Then $\forall n > 1\ \exists y_n \in L_n$:

$$||y_n|| = 1$$

and

$$\text{dist}\,(y_n, L_{n-1}) > 1/2$$

Proof. For Hilbert space it is obvious. Let P_n be orthogonal projection on L_n.

$$x^* := P_{n-1}x_n$$

$$y_n := \frac{x_n - x^*}{||x_n - x^*||}$$

Obviously $\operatorname{dist}(y_n, L_{n-1}) = 1$. For Banach a little bit more tricky: Take $x^* \in L_{n-1}$ such that $||x_n - x^*|| < 2\alpha$, where $\alpha = \operatorname{dist}(x_n, L_{n-1})$.

$$y_n := \frac{x_n - x^*}{||x_n - x^*||}$$

$\forall z \in L_{n-1}$

$$||y_n - z|| = \frac{1}{||x_n - x^*||}||x_n - (x^* + ||x_n - x^*||z)|| \geq \frac{1}{2\alpha}\alpha = 1/2$$

11 Vishik-Lax-Milgram Theorem

a) Continuous sesquilinear form B generates bounded operator A.
V - a Hilbert space, $B(\,\cdot\,,\,\cdot\,) : V \times V \to \mathbb{C}$ - sesquilinear, continuous.
Then necessarily: $|B(x,y)| \leq C||x||\,||y||$
Riesz $\Rightarrow$ $B(x,\,\cdot\,) =< Ax,\,\cdot\, >$

$$|< Ax, y >| = |B(x,y)| \leq C||x||\,||y||$$

$$y := Ax$$

$$||Ax||^2 =< Ax, Ax >\leq C||x||\,||Ax||$$

and

$$||Ax|| \leq C||x||$$

i. e. $A : V \to V$ is bounded.

b) If this continuous sesquilinear form B is V-elliptic (= ”positively” definite; in general, it is complex valued, hence ””) then $A : V \to V$ - isomorphism (and, in particular, surjective).

VLM Theorem: $|B(x,x)| \geq \alpha||x||^2$; $\alpha > 0 \quad \Rightarrow \quad A : V \to V$ - isomorphism (and A^{-1} is bounded).

Proof.

A injective: $||Ax||\,||x|| \geq |< Ax, x >| = |B(x,x))| \geq \alpha||x||^2$. Thus,

$$||Ax|| \geq \alpha||x|| \tag{11.1}$$ inverse

A surjective:
$R(A)$ - dense. $f \perp R(A), f \neq 0 \Rightarrow \forall x$ one has $0 =< Ax, f >= B(x,f)$. Take $x = f$?!
(11.1) implies that $R(A)$ is closed. Thus $R(A) = V$ and $||A^{-1}|| \leq \frac{1}{\alpha}$.

c) Existence of a weak solution to Neumann problem.
(*Follows from surjectivity of A from VLM Theorem*)

$$\begin{cases} (\Delta - \lambda)u = f(\in L_2(K)) & \text{in } K \\ \partial_n u = 0 \quad \text{on } \partial K \end{cases} \tag{11.2}$$ Neumann

What is a weak (= generalized) solution?
1)Integration by parts
Gauss theorem:

$$\Omega \subset \mathbb{R}^l : \int_\Omega \operatorname{div} \vec{E}\, dV = \int_{\partial\Omega} < \vec{n}, \vec{E} > \, dS$$

$$\partial_{x_k}(uv) = \operatorname{div}(0, \dots, 0, uv, 0, \dots, 0)$$
$$\vec{n} = (n_1, \dots, n_l)$$

Gauss formula implies:

$$\int_\Omega \partial_{x_k}(uv) dV = \int_{\partial\Omega} n_k uv dS$$

or

$\boxed{\int_\Omega u_{x_k} v\, dV = -\int_\Omega u v_{x_k}\, dV + \int_{\partial\Omega} n_k uv\, dS}$ - **formula of integration by parts**.
2) Now let u be classical solution of (11.2) and let $v \in C^\infty(K)$. Then

$$\int_K f\bar{v} dV = \int_K \Delta u\, \bar{v} dV - \lambda \int_K u\, \bar{v} = -\int_K \sum u_{x_i}\bar{v}_{x_i} + \int_{\partial K} \sum u_{x_i} n_i v\, dS - \lambda \int_K u\bar{v}\, dS =$$

$$= \int_K \sum u_{x_i}\bar{v}_{x_i}\, dV + \int_{\partial K} \frac{\partial u}{\partial n} \bar{v}\, dS - \lambda \int_K u\bar{v}\, dS =$$

$$-\int_K \sum u_{x_i}\bar{v}_{x_i}\, dV - \lambda \int_K u\bar{v}\, dV$$

Definition. $u \in W_2^1(K)$ is a generalized (weak) solution to (11.2) if $\forall v \in W_2^1(K)$
$\boxed{\int_K f\bar{v} dV = -\int_K \sum u_{x_i}\bar{v}_{x_i}\, dV - \lambda \int_K u\bar{v}\, dV}$
3) Introduce $B(\cdot,\cdot) : W_2^1(K) \times W_2^1(K) \to \mathbb{C}$:

$$B(u, v) = -\int_K \sum u_{x_i}\bar{v}_{x_i}\, dV - \lambda \int_K u\bar{v}\, dV$$

B is a continuous sesquilinear form.

Proposition 32. *For $\lambda \in \mathbb{C} \setminus \overline{\mathbb{R}_-}$ the form B is $W_2^1(K)$-elliptic.*

Proof

$$|B(u, u)|^2 = \Big| ||\nabla u||^2 + (\lambda_1 + i\lambda_2)||u||^2 \Big|^2 = (||\nabla u||^2 + \lambda_1 ||u||^2)^2 + \lambda_2^2 ||u||^4 =$$

$$||\nabla u||^4 + (\lambda_1^2 + \lambda_2^2)||u||^4 + 2||\nabla u||^2 \lambda_1 ||u||^2$$

WLOG $\lambda_1 < 0$ and $\lambda_2 \neq 0$. (If not then either obvious ($\lambda_1 > 0$) or in forbidden zone ($\lambda_1 < 0$ and $\lambda_2 = 0$)

$$2||\nabla u||^2|\lambda_1|\,||u||^2 \leq \epsilon^2||\nabla u||^4 + \frac{\lambda_1^2}{\epsilon^2}||u||^4$$

and

$$|B(u,u)|^2 \geq ||\nabla u||^4 + (\lambda_1^2 + \lambda_2^2)||u||^4 - \epsilon^2||\nabla u||^4 - \frac{\lambda_1^2}{\epsilon^2}||u||^4 =$$

$$||\nabla u||^4(1-\epsilon^2) + ||u||^2(\lambda_2^2 - (\frac{1}{\epsilon^2} - 1)\lambda_1^2) \geq$$

$$\alpha(||u||^4 + ||\nabla u||^4) \geq \frac{1}{2}\alpha(||u||^2 + ||\nabla u||^2)^2$$

if $\epsilon < 1$ and ϵ is close to 1. ($\epsilon^2 < 1$ and $\frac{1}{\epsilon^2} - 1$ is small).

4) Now **CENTRAL POINT**:

Proposition 33. *If $\lambda \in \mathbb{C} \setminus \overline{\mathbb{R}_-}$ then problem (Neumann 11.2) has a generalized solution from $W_2^1(K)$.*

Proof Due to VLM $B(u,v) = (Au,v)$ for $u,v \in W_2^1(K)$ and $A : W_2^1(K) \to W_2^1(K)$ - isomorphism. Linear functional $< f, \cdot >_{L_2(K)}$: $W_2^1(K) \to \mathbb{C}$ is continuous !!

$$(|< f,v >| \leq ||f||_{L_2}||v||_{L_2} \leq ||f||_{L_2}||v||_{W_2^1})$$

Riesz $\Rightarrow \exists U \in W_2^1$: $< f,v >_{L_2} = (U,v)_{W_2^1}$

A - isomorphism (and, in particular, SURJECTIVE). Thus, $\exists u_0 \in W_2^1$: $Au_0 = U$.

For this u_0

$$B(u_0,v) = (Au_0,v) = (U,v) =< f,v >_{L_2}$$

or

$$B(u_0,v) =< f,v >$$

for any $v \in W_2^1$. Thus, u_0 is a weak solution to (Neumann 11.2).

Remark. One can do the same with $f \in H^{-1}(K) = (W_2^1(K))^*$

12 Lemma used in the proof of Sears criterion

kid **Lemma 15.** *If $\Phi \in D'(R)$ and $\Phi' = 0$ then $\Phi = C$. If $\Phi'' = 0$ then $\Phi = Cx + D$.*

Proof of Lemma kid 15. Let $\Phi' = 0$. Choose $\phi_0 \in D(R)$ such that $\int_R \phi_0 \neq 0$. Let

$$C = \frac{< \Phi, \phi_0 >}{\int_R \phi_0}$$

For any $\phi \in D(R)$ one has

$$< \Phi - C, \phi >=< \Phi, \phi > - \frac{< \Phi, \phi_0 >}{\int_R \phi_0}\int_R \phi =< \Phi, \psi >$$

where $\psi = \phi - \frac{\phi_0 \int_R \phi}{\int_R \phi_0}$. One has $\int_R \psi = 0$ and therefore $\psi(x) = \frac{d}{dx} \int_{-\infty}^x \psi$ with

$$\int_{-\infty}^x \psi \in D(R) \quad (!!!!!)$$

Thus, $< \Phi, \psi >= 0$ and $\Phi = C$.

Let $\Phi'' = 0$ Then $\Phi' = C$ and $(\Phi - Cx)' = 0$. Thus $\Phi - Cx = D$.

Banach Algebras Approach to the Spectral Theorem:

Notes from Winter 2023 version of the course

13 Spectral Theorem for bounded self-adjoint operators

13.1 Banach algebra generated by A in BH.

Let $A \in \mathbf{BH}$, $A = A^*$. Consider $\mathcal{A}$ – the (closed) commutative Banach algebra (over $\mathbb{R}$!!!) generated by A in $\mathbf{BH}$.

Spectral Theorem

$$\mathcal{A} = C\left(\mathrm{Sp}\left(A\right)\right)$$

Let $\alpha I \leq A \leq \beta I$.
(Reminder: A, B - s. a., $A \geq B$ iff $A - B \geq 0$ i. e. $\forall x \quad < (A-B)x, x >\geq 0$.)
Define the map

$$\mathbb{R}[x] \ni p(x) \mapsto p(A) \in \mathcal{A}$$

Lemma 16. *If $p(x) \geq 0$ on $[\alpha, \beta]$ then $p(A) \geq 0$.*

Proof

Lemma 17. *Let A s. a., $A \geq 0$. Then there exists $\sqrt{A} \in \mathbf{BH}$, s. a., positive and commuting with A.*

Usually, this Lemma is proved via the Spectral Theorem, but an independent proof is also possible: $\sqrt{A}$ can be constructed via iterations. We skip this (or postpone).

Lemma 18. *Let A_1 and A_2 are s. a., positive, commuting. Then A_1A_2 is also s. a. and positive.*

Proof.

$$< A_1A_2x, x > = < A_1\sqrt{A_2}\sqrt{A_2}x, x > = < \sqrt{A_2}A_1\sqrt{A_2}x, x > = < A_1\sqrt{A_2}x, \sqrt{A_2}x > \geq 0$$

Now

$$p(x) = C\prod_i (x-\alpha_i)\prod_j(\beta_j - x)\prod_k(x-\gamma_k)^2\prod_l\left((x-\delta_l)^2+\epsilon_l^2\right)$$

where $C > 0$, α_i are the real roots from the left of α, β_j are the real roots from the right of β, γ_k are the roots from (α, β) (of even multiplicity!!); the last product corresponds to pairs of mutually conjugate complex roots. Clearly, $p(A) \geq 0$ by the preceeding Lamma.

Corollary.

$$||p(A)|| \leq \sup_{x\in[\alpha,\beta]}|p(x)|\,. \tag{13.1}$$ ineq

Proof. Let $S = \sup_{x\in[\alpha,\beta]}|p(x)|$. Consider

$$q_+(x) = S - p(x) \quad \text{and} \quad q_-(x) = S + p(x)$$

Both polynomials are positive on $[\alpha, \beta]$, therefore, $q_+(A \geq 0$ and $q_+(A) \geq 0$. This gives

$$\forall x \quad -S||x||^2 \leq < p(A)x, x > \leq S||x||^2$$

or

$$\sup_{||x||=1}| < p(A)x, x > | \leq S\,.$$

But for a bounded s. a. operator B

$$||B|| = \sup_{||x||=1}| < Bx, x > |$$

(skipped, see FA 1, postponed if unknown). Thus, $||p(A)|| \leq \sup_{x\in[\alpha,\beta]}|p(x)|$ as stated.

Thus, the map

$$p(x) \mapsto p(A)$$

extends to a (continuous) morphism of Banach algebras

$$i : C[\alpha, \beta] \to \mathcal{A}$$

(via Weierstrass Theorem).

Consider $I = \operatorname{Ker} i$ — a closed ideal in $C[a, b]$.

Define (notation will be justified later, it is in fact the spectrum of A)

$$\operatorname{Sp}(A) := Z(I) = \overline{\{x \in [a, b] : \forall f \in I \quad f(x) = 0\}}$$

Lemma 19.

$$I = \{f \in C[a, b] : f\Big|_{Z(I)} = 0\}$$

That is a general fact. Let K be a Hausdorf compact, and let I be a closed ideal in $C(K)$. Then $I = \{f \in C(K) : f|_{Z(I)} = 0\}$.

Reminder: Real version of Stone-Weierstrass: K – Hausdorf compact, $\mathcal{A} < C(K)$, $\mathcal{A}$ separates any two points and $1 \in \mathcal{A} \implies \overline{A} = C(K)$.

Proof. $Z(I)$ is compact. $K/Z(I)$ is again a (Hausdorf) compact.

$$C(K/Z(I)) = \{f \in C(K) : f|_{Z(I)} = \text{const}\}$$

The subalgebra $c\mathbf{1} + I$ of $C(K/Z(I))$ coincides with $C(K/Z(I))$ due to the real version of S.-W., so, I coincides with $\{f \in C(K) : f|_{Z(I)} = 0\}$.

Now define the map

$$C(\mathrm{Sp}\, A) \ni f \mapsto \tilde{f}(A) \in \mathcal{A}$$

where $\tilde{f}$ is a continuous extension of f to $[\alpha, \beta]$ with the same uniform norm. Clearly, the $\tilde{f}(A)$ is independent of the choice of the extension:

$$g|_{\mathrm{Sp}\,(A)} = 0 \implies g(A) = 0$$

Thus we have a morphism

$$j : C(\mathrm{Sp}\,(A)) \to \mathcal{A}$$

Theorem 5.

- *For $f \in C(\mathrm{Sp}(A))$ one has*

$$f \geq 0 \Leftrightarrow j(f) = f(A) \geq 0$$

- *$j : C(\mathrm{Sp}(A)) \to \mathcal{A}$ is an isomorphism of Banach algebras.*

*(***An alternative form of the SPECTRAL THEOREM***)*

Proof. 1) Let $f(A) \geq 0$. We have to prove that $f \geq 0$ on $\mathrm{Sp}\,(A)$.

Let $c \in \mathrm{Sp}(A), f(c) < 0$. Let ξ be a positive cut-off function with support near c. Then ξf is negative everywhere and $-\xi(A)f(A) \geq 0$. But $f(A)$ is positive and $\xi(A)$ is positive, so $\xi(A)f(A) \geq 0$. Thus, $f(A)\xi(A) = 0$ and $f\xi \in \mathrm{Ker}\, i$ and, therefore, $f\xi|_{\mathrm{Sp}(A)} = 0$ which gives a contradiction.

2)$b := ||f(A)||$, then

$$bI \pm f(A) \geq 0$$

and, therefore,

$$b \pm f(t) \geq 0$$

on $\mathrm{Sp}(A)$ and

$$\sup_{\mathrm{Sp}(A)} |f| \leq b = ||f(A)||$$

□

Finally, we identify $\mathrm{Sp}\,(A)$ *with the spectrum of* A.

Temporarily denote the spectrum of A by $\sigma(A)$. 1)$\sigma(A) \subset \mathrm{Sp}(A)$ or $\mathrm{Sp}(A)^c \subset \sigma(A)^c$.

Let $\xi \in \mathbb{R}$ does not belong to $\mathrm{Sp}(A)$. then $t \mapsto \xi$ is invertible on $\mathrm{Sp}(A)$ and $A - \xi I$ is also invertible.

2)$\mathrm{Sp}(A) \subset \sigma(A)$.

Let $\xi \in \mathrm{Sp}\,(A)$. Let g be a positive function equal to N in $1/n$-vicinity of ξ and $g(t) = 1/|t-\xi|$ outside this vicinity.

Assume that $A - \xi I$ is invertible. Let $B = (A - \xi I)^{-1}$.

$$B(A - \xi I) = (A - \xi I)B) = I$$

Since $|(t-\xi)g(t)| \leq 1$, one has

$$||(A - \xi I)g(A)|| \leq 1$$

and

$$||g(A)|| = ||B(A - \xi I)g(A)|| \leq ||B||$$

But $||g(a)|| = N$, which gives a contradiction.

Exercise. Show that the C^*-algebra generated by I and A in **BH** is isomorphic to $C_{\mathbb{C}}(\mathrm{Sp}(A))$.

13.2 On projection operators.

G - closed subspace H, $H = G \oplus F$, $h = g + f$, $h \mapsto g$ is the orthogonal projection $P_G h$.

Lemma 20. *Let $P : H \to H$ be a linear operator. Then $P^2 = P$ & $P^* = P$ iff $P = P_G$ for some closed $G < H$.*

Proof. $G := \{h \in H : Ph = h\}$.

Clearly, $\forall h \in H \;\; P_G h \in G, Ph \in G$. Thus, one has to prove that $((P_G - P)h, g) = 0$ for any $g \in G$. That immediately follows from $P = P^*$ and $P_G^* = P_G$.

Properties

- $$||P|| = 1, \quad (Px, x) \geq 0\,,$$

  trivial

- $P \neq 0 \implies \sigma(P) = \{0, 1\}$.

  Directly: $(P - \lambda I)^{-1} = \frac{1}{1-\lambda}P - \frac{1}{\lambda}(I - P)$ (can be also obtained via Neumann series).

- Let $L, M < H$. Then $P_L P_M = P_M P_L = 0$ iff $L \perp M$.

- $P_L + P_M$ is a projector iff $L \perp M$.

- $P_M P_L = P_L P_M = P_M$ iff $M < L$.

- $P_L - P_M$ is a projector iff $M < L$.

- $P_M P_L$ is a projector iff $P_M P_L = P_L P_M$ (and in this case $P_L P_M = P_{L\cap M}$).

  Proof. $\Rightarrow$:

  $P_1P_2 = (P_1P_2)^* = P_2^*P_1^* = P_2P_1$;

  $\Leftarrow$:

  $(P_1P_2)^2 = P_1P_2P_1P_2 = P_1^2P_2^2 = P_1P_2$; $\quad (P_1P_2)^* = P_2^*P_1^* = P_2P_1 = P_1P_2$.

  $P_1P_2h \in H_1 =: M$, $P_2P_1 \in H_2 =: L$ and $P_1P_2 : H \to M \cap L$.

13.3 Extension of the map $C[\alpha, \beta] \mapsto \mathcal{A} \subset \mathbf{BH}$.

Preliminary Remark. One can derive the usual form of the Spectral Theorem (i. e. unitary equivalence to the operator of multiplication) from its Alternative form stated above. This will be done a little bit later. After that the map $C[\alpha, \beta] \mapsto \mathcal{A} \subset \mathbf{BH}$ and its extension (say, to characteristic functions of a measurable set) become obvious (multiplication by a bounded measurable function is a bounded operator in L_2).

Lemma 21. *Let A_n - s. a., $A_n \geq \alpha I$, $A_n \geq A_{n+1}$. Then $\forall v \in H$ $\exists \lim A_n v =: Av$ and A is bounded and s. a.*

Proof. Clearly $(A_n v, v)$ converges for any $v \in H$. Polarization identity implies $(A_n v, w)$ converges for all $v, w \in H$. Define an antilinear functional

$$\lambda_v(w) = \lim(A_n v, w)$$

$$||A_1||^2 = \sup((A_1 x, x) \geq \sup((A_n x, x) = ||A_n||$$

Thus, $||A_n|| \leq C$ and, therefore,

$$|\lambda_v(w)| \leq \mathrm{const}||w||$$

and there is an operator A such that

$$(Av, w) = \lim(A_n v, w)$$

It is s. a. since $(A_n v, w) = (v, A_n w)$ and bounded because the adjoint is closed.

Let us prove that $s - \lim A_n = A$. One has

$$||A_n v - Av||^2 = ||\sqrt{A_n - A}\sqrt{A_n - A}v||^2 \leq$$

$$\leq ||\sqrt{A_n - A}||^2(\sqrt{A_n - A}v, \sqrt{A_n - A}v) = ||A_n - A||((A_n - A)v, v)$$
$$\leq (C + ||A||)((A_n - A)v, v) \to 0$$

(the relation $||B||^2 = ||BB^*||$ and the self-adjointness of $\sqrt{A_n - A}$ was used).

Remark. This proof, taken from Appendix to Lang's "$SL_2(R)$" is less tricky than the one from Riesz–Sz-Nagy book. However the final step was missed by Lang (he proved only weak convergence) and the trick with square roots is taken from Murphy book on C^*-algebras (where the above Lemma is called "Vigier Theorem").

Lemma 22. *Let $f \geq -C$ on $\mathrm{Sp}(A)$ and let f_n are continuous on $\mathrm{Sp}(A)$, $f_n \geq f_{n+1}$ on $\mathrm{Sp}(A)$ and $f_n \to f$ on $\mathrm{Sp}(A)$ point-wise.*

Then $\lim f(A_n)$ (given by the previous lemma) is independent of the choice of the sequence $\{f_n\}$.

Proof. Let g_n, h_n be such sequences. Fix k and ϵ. Then $g_n \leq h_k + \epsilon$ for sufficiently big n. Thus $g_n(A) \leq h_k(A) + \epsilon I$ and

$$\lim g_n(A) \leq h_k(A) + \epsilon I$$

and, therefore,

$$\lim g_n(A) \leq h_k(A)$$

send $k \to \infty$. Change g and h.

So, the map $f \mapsto f(A)$ can be extended to functions that are bounded from below and can be represented as point-wise limits of monotonously decreasing sequences of continuous functions.

In particular, $\mathbf{1}_{x \leq \lambda}$ are such. Let $E(\lambda) = \mathbf{1}_{x \leq \lambda}(A)$.

Since $\mathbf{1}_{x \leq \lambda}(A)\mathbf{1}_{x \leq \lambda}(A) = \mathbf{1}_{x \leq \lambda}(A)$, $E(\lambda)^2 = E(\lambda)$ and $E(\lambda)$ is a projection in H.

13.4 Resolution of identity (aka spectral family of projections)

Clearly, $\mu \leq \lambda \implies E(\mu) \leq E(\lambda)$. Let $\alpha I \leq A \leq \beta I$, then $E(\lambda) = 0$ for $\lambda \leq \alpha$ and $E(\lambda) = I$ for $\lambda \geq \beta$.

Lemma 23. *(Strong right continuity)*

$$s - \lim_{\epsilon \to 0+} E(c + \epsilon) = E(c)$$

Proof. Clearly, $E(c+\epsilon) - E(c)$ is a projection, therefore,

$$||((E(c+\epsilon) - E(c))v||^2 = ((E(c+\epsilon) - E(c))v, v)\,.$$

So one has to prove that

$$(E(c+\epsilon)v, v) \to (E(c)v, v)$$

as $\epsilon \to 0+$. Let h_ϵ continuously extends

$$\begin{cases} 1, & x \leq c \\ 0, & x \geq c + \epsilon \end{cases}$$

to $\mathbb{R}$ (say, via linear interpolation) and h_δ similarly extends

$$\begin{cases} 1, & x \leq c + \epsilon \\ 0, & x \geq c + \epsilon + \delta \end{cases}$$

Clearly, $h_\delta \to h_\epsilon$ uniformly as $\delta \to 0$. Therefore, $h_\delta(A) \to h_\epsilon(A)$ in the operator norm as $\delta \to 0$. (Continuous functional calculus for A). One has

$$\mathbf{1}_{x \leq c} \leq \mathbf{1}_{x \leq c+\epsilon} \leq h_\delta$$

and

$$h_\delta \leq h_\epsilon + \epsilon$$

for sufficiently small δ. So,

$$\mathbf{1}_{x\leq c} \leq \mathbf{1}_{x\leq c+\epsilon} \leq h_\epsilon + \epsilon$$

and, since $(h_\epsilon(A)v, v) \to (\mathbf{1}_{x\leq c}(A)v, v)$ as $\epsilon \to 0+$, one gets the statement of the Lemma.

Spectral inequality

Lemma 24. *Let $b \leq c$. Then*

$$b(E(c) - E(b)) \leq A(E(c) - E(b)) \leq c(E(c) - E(b)) \tag{13.2}$$ Spineq

Proof. The inequality

$$b(E(c) - E(b)) \leq A(E(c) - E(b))$$

follows from the inequality

$$bI\Big|_{\ker E(b)} \leq A\Big|_{\ker E(b)} \tag{13.3}$$ eq1

In fact, let $h \in H$, $h = u + v$ with $u \in \ker E(b)$, $v \in (\ker E(b))^\perp$.

Then $(E(c) - E(b))v = 0$. Indeed, since $E(b)(E(c) - E(b)) = E(b) - E(b) = 0$, one has $\forall h \in H \ \ (E(c) - E(b))h \in \ker E(b)$ and

$$\forall g \in H \quad ((E(b) - E(c))v, g) = (v, (E(b) - E(c))g) = 0\,.$$

On the other hand, if $v \in \ker E(b)$ then $E(c)v \in \ker E(b)$, too.

Similarly, the inequality

$$A(E(c) - E(b)) \leq c(E(c) - E(b))$$

follows from the inequality

$$A\Big|_{E(c)H} \leq cI\Big|_{E(c)H}\,. \tag{13.4}$$ eq2

(since $E(c)E(b) = E(b)$ and, therefore, $(E(c) - E(b))H \subset E(c)H$, this is obvious).

Now consider the product, f_b, of two functions:

- $x - b$
- $\mathbf{1}_{x>b}$

Clearly,

$$(A - bI)(I - E(b)) = f_b(A) \geq 0$$

and, (eq1 13.3) follows.

Similarly, the product, g_c of $(x - c)$ and $\mathbf{1}_{x\leq c}$ is negative. Therefore, $(A - cI)E(c)$ is a negative operator.

Theorem 6. *(Lorch)*

$$\exists s - \lim_{\epsilon\to 0+}(E(c) - E(c-\epsilon)) = Q_c$$

and Q_c is the projection on $\{v \in H \ : \ Av = cv\}$

Proof. From spectral inequality one gets

$$(c-\epsilon)(E(c) - E(c-\epsilon)) \le A(E(c) - E(c-\epsilon)) \le c(E(c(-E(c-\epsilon))$$

or

$$-\epsilon(E(c) - E(c-\epsilon)) \le (A - cI)(E(c) - E(c-\epsilon)) \le 0$$

This gives (use $||B|| = \sup|(Bx, x)|$)

$$||(A - cI)(E(c) - E(c-\epsilon)|| \le \epsilon \tag{13.5}$$ ocenka

On the other hand Vigier theorem guarantees that

$$\forall v \in H \ \ \exists \lim_{\epsilon\to 0+}(E(c) - E(c-\epsilon))v =: w$$

(more elementary (no Vigier, only basic definitions - exercise):

Lemma 25. *$\{P_i\}_{i=1}^{\infty}$ - projections, $P_i \ge P_{i+1}$. Then $\exists s - \lim P_i = Q$ and Q is a projection.*

)
and (ocenka 13.5) implies

$$((A - cI)w = 0$$

or $Aw = \lambda w$.

Thus, the projectiion Q_c sends H to $\ker(A - cI)$.

It remains to show that

$$Q_c\Big|_{\ker,(A-cI)} = Id \tag{13.6}$$ triv

One can consider all the operators involved $(A, I, E(b), E(c))$ as elements of $B(\ker(A - cI))$. Then $A = cI$. Since $\mathrm{Sp}(A) = \{c\}$, $1_{x\le c} = 1$ on $\mathrm{Sp}(A)$ and $E(c) = I$ and $E(b) = 0$ if $b < c$. Thus, (triv 13.6) holds true.

13.5 Integral representation fo A

$$\alpha I \le A \le \beta I$$

$$\mu_0 < \alpha < \mu_1 < \mu_2 < \cdots < \mu_{n-1} < \beta < \mu_n$$

$$\sup_k(\mu_{k+1} - \mu_k) \le \epsilon$$

Spectral inequality gives

$$\mu_{k-1}(E(\mu_k) - E(\mu_{k-1})) \le A(E(\mu_k) - E(\mu_{k-1})) \le \mu_k(E(\mu_k) - E(\mu_{k-1}))$$

Summing up one gets

$$\sum_{k=1}^{n} \mu_{k-1}(E(\mu_k) - E(\mu_{k-1})) \leq A \leq \sum_{k=1}^{n} \mu_k(E(\mu_k) - E(\mu_{k-1}))$$

and

$$\sum_{k=1}^{n} \mu_k(E(\mu_k) - E(\mu_{k-1})) - \sum_{k=1}^{n} \mu_{k-1}(E(\mu_k) - E(\mu_{k-1})) \leq \epsilon I$$

Thus,

$$||A - \sum_{k=1}^{n} \lambda_k(E(\mu_k) - E(\mu_{k-1}))|| \leq \epsilon$$

if $\mu_k \leq \lambda_k \leq \mu_k$ and

$$A = \int_{-\infty}^{+\infty} \lambda dE(\lambda) = \int_{\alpha-}^{\beta} \lambda dE(\lambda)$$

(as projection Stiltjes integral)

13.6 Equivalence with multiplication form of the S. T.

First let us pass to complex-valued functions. Extend i to a $*$-morphism

$$C_{\mathbb{C}}[\alpha, \beta] \to \overline{\{\sum^{<\infty} a_k A^k, a_k \in \mathbb{C}\}} =: \mathcal{A}_{\mathbb{C}} < \mathbf{BH}$$

in an obvious way. One has

$$||i(f + \sqrt{-1}g)|| \leq ||i(f)|| + ||i(g)|| \leq \sup_{|} f| + \sup|g| \leq 2\sup|f + \sqrt{-1}g|$$

Let $f \in H$, consider $H_f = \mathcal{A}_{\mathbb{C}} f = \overline{\{\sum^{<\infty} a_k A^k, a_k \in \mathbb{C}\}}$. In case $\mathcal{A}f = H$ the vector f is called cyclic. In general,

$$H = \oplus_{k=1}^{\infty} H_{f_k} \tag{13.7}$$ dsum

(exercise - the same me thod as for resolvent way).

Consider the (bounded, positive) functional:

$$C[\alpha, \beta] \in \phi \mapsto (\phi(A)f, f)\,.$$

By R-M Theorem

$$(\phi(A)f, f) = \int_{\mathbb{R}} \phi d\mu_f$$

for some finite measure μ on $\mathbb{R}$ (in fact with *supp* on $\sigma(A)$).

In particular,

$$(A^n f, A^m f) = (A^{n+m} f, f) = \int x^{n+m} d\mu_f(x) =< x^n, x^m >_{L_2(\mathbb{R}, d\mu_f)}$$

So, we get an isometry

$$U : H_f \to L_2(\mathbb{R}, d\mu_f)$$

Clearly, action of A is unitary equivalent to multiplication by x. Finally, using (dsum 13.7), one constructs the unitary equivalence

$$\mathcal{U} : H \to \oplus_{k=1}^{\infty} L_2(\mathrm{SpA}; d\mu_k)$$

with $\mathcal{U}A\mathcal{U}^{-1}$ acting as M_x.

Let M be the disjoint union $\cup_k \mathrm{SpA}$ with measure μ whose restriction on k-th component is μ_k. Let $\hat{A} : M \to \mathbb{R}$ be the function defined by $\hat{A}(p) = x(p)$ if p is in k-th component of M.

If f is a *bounded* measurable (say, Borel) function on $\mathrm{Sp}(A)$ then define

$$f(A) := \mathcal{U}^{-1} f(\hat{A}) \mathcal{U}$$

Exercise: f_j - uniformly bounded (measurable, or, what suffices, continuous). f_j converges pointwise to f. Then

$$\exists s - lim f_j(A) = f(A)\,.$$

Hint: this immediately follows from Lebesgue dominated convergence theorem.

Thus, we arrive at the extended functional calculus as it was constructed above.

13.7 Spectral Theorem for commuting operators

With a very small effort one can prove the generalization of ST for s. a. bounded $A_1, \ldots, A_n$, such that $[A_i, A_j] = 0$: there is an unitary equivalence $\mathcal{U} : H \to L_2(M, \mu)$ such that $\mathcal{U}A_k\mathcal{U}^{-1}$ are M_{f_k} with some (bounded) measurable real-valued f_k. (In fact M is the product of $\mathrm{Sp}(A_i)$.)

13.8 Derivation of the ST for unbounded s. a. operators

Recall that operator B is called normal if $BB^* = B^*B$. If B is normal then

$$B = A_1 + iA_2$$

with self-adjoint commuting A_1, A_2. In fact,

$$B = \frac{B + B*}{2} + i\frac{B - B^*}{2i}$$

Thus, using the ST for commuting A_1, A_2, one gets the spectral theorem for bounded normal operator: it is unitary equivalent to operator of multiplication $M_{f_1+if_2}$ by a complex-valued function.

Now let $A : H \to H$ be s.a, unbounded. According to Lemma 4 §3.1.2 (page 31) of the Main Course, $N := (iI - A)^{-1}$ exists and is bounded.

Lemma 26. *The operator N is normal.*

Proof. One has

$$[(iI - A)^{-1}]^* = [(iI - A)^*]^{-1} = (-iI - A)^{-1}$$

and

$$(iI - A)^{-1}[(iI - A)^{-1}]^* = (iI - A)^{-1}(-iI - A)^{-1} = [(-iI - A)(iI - A)]^{-1} =$$

$$[(iI - A)(-iI - A)]^{-1} = (-iI - A)^{-1}(iI - A)^{-1} = [(iI - A)^{-1}]^*(iI - A)^{-1}$$

Therefore, $(iI - A)^{-1}$ is unitary equivalent to the multiplication operator M_ψ in $L_2(M, d\mu)$ with some complex-valued ψ. Let $\phi = i - \frac{1}{\psi}$. Then A is unitary equivalent to M_ϕ. Since A is self-adjoint, ϕ should be a. e. real. $\square$

Friedrichs extension & around.

§1. Abstract setting

general construction: $\begin{cases} \text{Hilbert spaces } H_1, H_0 \\ H_1 \subset H_0,\ H_1 \text{ dense in } H_0 \\ \forall u \in H_1\ \ \|u\|_0 \le \|u\|_1 \\ \text{(i.e. } \subset \text{ is continuous)} \end{cases}$ $\rightsquigarrow$ a self-adjoint (unbounded) operator in H_0.

(1.1.) H_{-1}, $J: H_1 \to H^{-1}$

$H_{-1} := H_1^*$ (bounded linear f-ls on H_1); $\langle\ ,\ \rangle$ - pairing of H_{-1} & H_1

$\langle f, h_1 \rangle \equiv (J^{-1} f, h_1)_1$, where $J^{-1}: H_{-1} \to H_1$ linear bijection

$(f, g)_{-1} := (J^{-1} f, J^{-1} g)_1$, H_{-1} - Hilbert, J, J^{-1} - isometries

(1.2) $H_0 \overset{j}{\hookrightarrow} H_{-1}$

$j: H_0 \to H_{-1}$ $\qquad \langle j h_0, h_1 \rangle := (h_0, i h_1)_0$

where $i: H_1 \to H_0$ ($\underset{\text{inclusion}}{\subset}$)

$|\langle j h_0, h_1 \rangle| \le \|h_0\|_0 \|i h_1\|_0 \le \|h_0\|_0 \|h_1\|_1$

Two main formulae for the future:

$\langle f, h_1 \rangle = (J^{-1} f, h_1)_1$ ★ $\quad \Big\| \quad$ $\langle j h_0, h_1 \rangle = (h_0, i h_1)_0$ $*$

1.3 Diagram

$$\begin{tikzcd} H_1 \arrow[r, "i"] \arrow[dr, dashed, bend right, "J"'] & H_0 \arrow[d, "j"] \\ & H_{-1} \arrow[ul, "J^{-1}"'] \end{tikzcd}$$

Not Commutative !!

$h_1 \xrightarrow{i} ih_1 \xrightarrow{j} (ih_1, \cdot)_0 : H_1 \to \mathbb{C}$

$h_1 \to Jh_1$

$\langle Jh_1, \underset{H_1}{\underset{\cap}{\bullet}} \rangle \overset{*}{=} (J^{-1}Jh_1, \bullet)_1 = (h_1, \bullet)_1$

1.4 $ji(H_1)$ dense in H_{-1}

(& therefore, $j(H_0)$ dense in H_{-1})

Proof Let $z \in H_{-1} : \forall h_1 \in H_1 \quad (z, j(i(h_1)))_{-1} = 0$

$$0 = (z, jih_1)_{-1} = (J^{-1}z, J^{-1}jih_1)_1 \overset{*}{=} \overline{\langle jih_1, J^{-1}z \rangle} \overset{*}{=} \overline{(ih_1, iJ^{-1}z)_0} \Rightarrow iJ^{-1}z = 0 \Rightarrow z = 0.$$

iH_1 dense in H_0

1.5 Domain of the future s.a. operator.

$$\begin{tikzcd} H_1 \arrow[r, "i"] & H_0 \arrow[d, "j"] \\ & H_{-1} \arrow[ul, "J^{-1}"] \end{tikzcd} \qquad \mathcal{D} := i\, J^{-1} j(H_0) \subset H_0$$

Density of $\mathcal{D}$ in H_0:

$j(H_0)$ – dense in H_{-1} (Corollary ↑) $\Rightarrow$ $J^{-1}j(H_0)$ – dense in H_1

$$\left.\begin{array}{l} J^{-1}j(H_0) \text{ – dense in } H_1 \\ i(H_1) \text{ – dense in } H_0 \end{array}\right\} \Rightarrow i\, J^{-1} j(H_0) \text{ – dense in } H_0$$

(exercize – trivial).

1.6 Operator J_0 with domain $\mathcal{D}$.

$J_0 : \mathcal{D} \to H_0$

$J_0 = j^{-1} J\, i^{-1}$

$$\begin{tikzcd} H_1 \arrow[r, "i"] \arrow[dr, "J"'] & H_0 \arrow[d, "j"] \\ & H_{-1} \end{tikzcd}$$

$J_0 : H_0 \supset \mathcal{D} \to H_0$

$\mathcal{R}(J_0) = H_0$. $\left(j^{-1} J i^{-1} i J^{-1} j H_0 = H_0 \right)$

$\mathcal{D}$ is dense in H_0.

J_0 – injection – clear.

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1.7 J_0 is self-adjoint

Consider $J_0^{-1} : H_0 \longrightarrow \mathcal{D} \subset H_0$

We will prove that $\forall u \in H_0$ $\quad (J_0^{-1}u, u)_0 \in \mathbb{R}$ $\Longrightarrow J_0^{-1}$-symmetric

$\Rightarrow$ ~~closed~~ $\Rightarrow$ s.a. ‖ Therefore, J_0^{-1} is s.a.
$\searrow$ bounded

$$(J_0^{-1}u, u)_0 = (J_0^{-1}u, J_0 J_0^{-1} u)_0 =$$

$$\boxed{J_0^{-1} u \in \mathcal{D} \Longrightarrow J_0^{-1}u = i J^{-1} j h_0, \quad h_0 \in H.}$$

$$= (i J^{-1} j h_0, J_0 i J^{-1} j h_0)_0 = \overline{(J_0 i J^{-1} j h_0, i J^{-1} j h_0)_0} =$$

$$\overset{*}{=} \overline{\langle j J_0 i J^{-1} j h_0, J^{-1} j h_0 \rangle} \overset{\star}{=} \overline{(J^{-1} j \underbrace{J_0}_{j^{-1} J i^{-1}} i J^{-1} j h_0, J^{-1} j h_0)_1} \quad = id$$

$$= (J^{-1} j h_0, J^{-1} j h_0)_1 \geq 0 \; (\in \mathbb{R})$$

§2) $A: H_0 \to H_0$, symmetric, $\forall x \in \mathcal{D}(A)$ $(Ax, x)_0 \geq (x,x)_0$

$\mathcal{D}(A)$ $\quad (u,v)_1 = (Au, v)$; H_1 – completion of $\mathcal{D}(A)$ in $(\cdot,\cdot)_1^{1/2} = |||\cdot|||$

$i_1: \mathcal{D}(A) \to H_0$
extends to $i: H_1 \to H_0$
$i(H_1)$ is dense in H_0

So, we are in our abstract setting.

$\boxed{J_0 \text{ is a extension of } A}$ (self-adjoint)

2.1. Why i is injective? (Replaces somewhat tricky explanation from my old lectures)

$\forall u,v \in \mathcal{D}(A)$ $(u,v)_1 = (u, Av)_0 = (i_1 u, Av)_0$

This is true for all $u \in H_1$ and $v \in \mathcal{D}(A)$

$i_1 u = 0 \Rightarrow u \perp_{H_1} \mathcal{D}(A) \Rightarrow u = 0$ in H_1. ($\mathcal{D}(A)$ is dense in H_1 by definition)

2.2. Why $\mathcal{D}(A) \subset \mathcal{D}$

$x \in \mathcal{D}(A) \Rightarrow Ax \in H_0$

$\forall h \in \mathcal{D}(A)$ $(Ax, h)_0 = (\overset{i^{-1}x}{x}, h)_1$

$\langle jAx, h \rangle \overset{☆}{=} (J^{-1} jAx, h)_1$

for $x \in \mathcal{D}(A)$, $H_1 \ni x$ & $ix \in H_0$ is the same

$\Rightarrow \boxed{i^{-1} x = J^{-1} j Ax}$ ☆☆

$x = \underbrace{i J^{-1} j}\, Ax \in i J^{-1} j(H_0) = \mathcal{D}$

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2.3. Why $A|_{\mathcal{D}(A)} = J_0|_{\mathcal{D}(A)}$

$J_0 = j^{-1} J \underbrace{i^{-1}}_{\text{"id on } \mathcal{D}(A)\text{"}}$

$\boxed{x \in \mathcal{D}(A)}$

$J_0 x = \underbrace{j^{-1} J i^{-1}}_{J_0} \underbrace{i J^{-1} j A x}_{x} = Ax.$ ∎

A - strictly positive, symmetric

Plain language:

$H_1 \xrightarrow[i]{} H_0$

$A : \overset{\mathcal{D}(A)}{\overset{\cap}{H_0}} \to H_0$

H_1 - energetic space (completion of $\mathcal{D}(A)$ in $\|\cdot\|_1$)

$\mathcal{D}(\hat{A}) = i\left(\{e \in H_1 : (e, \cdot)_1 \text{ is bounded in } H_0\}\right)$

i.e. $(e, \tilde{e})_1 \le c \|\tilde{e}\|_0 \quad \forall \tilde{e} \in \mathcal{D}(A)$

$e \in \mathcal{D}(\hat{A}) \Rightarrow (e, \cdot)_1 = (h, \cdot)_0$ for $h \in H_0$ (Riesz)

$\hat{A} e := h.$

Exercise:

Prove:

$\hat{A} = J_0$

Friedrichs extension