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INSTITUTE OF PHYSICS PUBLISHING JOURNAL OF PHYSICS A: MATHEMATICAL AND GENERAL

J. Phys. A: Math. Gen. **39** (2006) 8997–9013 doi:10.1088/0305-4470/39/28/S14

Bergman tau-function: from random matrices and Frobenius manifolds to spaces of quadratic differentials

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Received 28 October 2005, in final form 2 May 2006
Published 27 June 2006
Online at stacks.iop.org/JPhysA/39/8997

Abstract

We discuss the notion of the Bergman tau-function on Hurwitz spaces, spaces of abelian and quadratic differentials on Riemann surfaces. The Bergman tau-function on Hurwitz spaces is nothing but isomonodromic tau-function of Hurwitz Frobenius manifolds; it also appears in genus one contribution to partition function of Hermitian two-matrix model. The Bergman tau-function on spaces of abelian and quadratic differentials appears in the problem of holomorphic factorization of determinants of Laplacians in flat metrics with conical singularities over Riemann surfaces.

PACS number: 02.10.Yn

1. Introduction and some preliminaries

In this short survey (which is an extended version of two talks of the authors given at CRM Program on Random Matrices), we discuss some applications and further generalizations of the Bergman tau-function. This object was first introduced in the context of Hurwitz spaces [15] (the moduli spaces of meromorphic functions on compact Riemann surfaces) and later was found to be universal: it appears in the theory of Frobenius manifolds, theory of isomonodromic deformations, Hermitian two-matrix model and spectral theory of Laplacians on Riemann surfaces. Here we generalize the Bergman tau-function to the case of the moduli spaces of meromorphic quadratic differentials. This generalization is used in the explicit calculation of the determinants of the Laplacians corresponding to Strebel metrics of finite volume, i.e. flat conical metrics defined by the modulus of a meromorphic quadratic differential with at most simple poles.

To introduce the notion of the Bergman tau-function we need some technical tools of the analysis on compact Riemann surfaces, in particular, we assume that the reader is familiar with the prime form, the canonical meromorphic bidifferential and properties of projective connections (see [9, 28]); the so-called Bergman projective connection will be used in all our forthcoming constructions. Here we recall some basic definitions.

Let $\mathcal{L}$ be a compact Riemann surface of genus g which is Torelli marked (i.e. a canonical symplectic basis $\{a_\alpha, b_\alpha\}_{\alpha=1}^g$ in $H_1(\mathcal{L}, \mathbb{Z})$ is chosen). Then one can introduce the canonical (i.e. depending only on Torelli marking of the surface) meromorphic bidifferential $\mathbf{w}(P, Q)$. We recall that $\mathbf{w}(P, Q)$ has zero a-periods with respect to both its arguments, is nonsingular everywhere except the diagonal $P = Q$ and is subject to the following asymptotics as $P \to Q$:

$$\mathbf{w}(P, Q) = \left(\frac{1}{(x(P) - x(Q))^2} + \frac{1}{6} S_B(x(P)) + o(1)\right) \mathrm{d}x(P)\, \mathrm{d}x(Q), \tag{1.1}$$

where $x(P)$ is a local parameter near Q and S_B is a projective connection.

We recall that a quantity S is called a projective connection if it behaves as follows under the change of the local parameter $x = x(y)$:

$$S(y) = S(x)\left(\frac{\mathrm{d}x}{\mathrm{d}y}\right)^2 + \{x, y\}, \tag{1.2}$$

where $\{x, y\}$ is the Schwarzian derivative.

The projective connection S_B from (1.1) is called *the Bergman projective connection*.

Another important example of a projective connection can be obtained as follows. Let ω be a meromorphic one-differential (one can even afford multi-valued one-differentials having constant multiplicative twists along a and b-cycles). Then one can define a projective connection S_ω setting

$$S_\omega(x(P)) = \left\{\int^P \omega, x(P)\right\}, \tag{1.3}$$

for any local parameter $x(P)$. Due to the properties of the Schwarzian derivative the right-hand side of (1.3) depends neither on the choice of the starting point of the integration, nor the contour of integration, nor on the choice of the branch of differential ω (in the case when the latter is multivalued).

It follows from definition (1.2) that the difference of two projective connections is a quadratic differential.

Choose a basepoint on the Riemann surface $\mathcal{L}$ and $2g$ loops starting at this basepoint and forming a canonical homotopy basis for $\mathcal{L}$. Dissecting $\mathcal{L}$ along these loops one gets the fundamental polygon $\widehat{\mathcal{L}}$. Let $\{v_\alpha\}_{\alpha=1}^g$ be the basis of normalized ($\oint_{a_\alpha} v_\beta = \delta_{\alpha\beta}$) holomorphic differentials and let $\mathbf{B} = \| \oint_{b_\alpha} v_\beta \|$ be the corresponding matrix of b-periods.

For any point P inside the fundamental polygon introduce the vector of Riemann constants:

$$K_\alpha^P = \frac{1}{2} + \frac{1}{2}\mathbf{B}_{\alpha\alpha} - \sum_{\beta=1, \beta\neq\alpha}^{g} \oint_{a_\beta} \left(v_\beta \int_P^x v_\alpha\right) \tag{1.4}$$

where the interior integral is taken along a path which does not intersect $\partial\widehat{\mathcal{L}}$.

For $g \geqslant 1$, define

$$\mathcal{C}(P) := \frac{1}{\mathcal{W}[v_1, \ldots, v_g](P)} \sum_{\alpha_1, \ldots, \alpha_g = 1}^{g} \partial^g_{\alpha_1 \cdots \alpha_g} \theta(K^P) v_{\alpha_1} \cdots v_{\alpha_g}(P), \tag{1.5}$$

where

$$\mathcal{W}(P) := \det_{1\leqslant\alpha,\beta\leqslant g} \left\| v_\beta^{(\alpha-1)}(P) \right\|$$

is the Wronskian of basic holomorphic differentials at the point P; K^P is the vector of Riemann constants corresponding to the basepoint P, θ is the theta-function. The $\mathcal{C}(P)$ is a multi-valued holomorphic $g(1-g)/2$-differential on $\mathcal{L}$ depending on the choice of the fundamental polygon $\widehat{\mathcal{L}}$. This differential (which is an essential ingredient of the explicit formula for the Mumford measure on the moduli space of Riemann surfaces of genus g) was introduced and extensively studied in [10].

2. Bergman tau-function on Hurwitz spaces

2.1. Hurwitz spaces

We define Hurwitz spaces as moduli spaces of pairs $(\mathcal{L}, \lambda)$, where $\mathcal{L}$ is a compact Riemann surface of genus g and λ is a meromorphic function on $\mathcal{L}$ with fixed multiplicities of zeros and poles of the meromorphic differential $d\lambda$. For brevity, we shall restrict ourselves to the case of the Hurwitz spaces $H_{g,N}(k_1, \ldots, k_l)$, when the differential $d\lambda$ must have l poles of multiplicities $k_1+1, k_2+1, \ldots, k_l+1$ and $2g-2+l+N$ *simple* zeros (note that $N = k_1+\cdots+k_l$ coincides with the degree of the function λ).

Instead of speaking about meromorphic functions one can use the equivalent language of branched coverings of the Riemann sphere $\mathbb{C}P^1$. This leads to the definition of the Hurwitz space $H_{g,N}(k_1, \ldots, k_l)$ as the set of equivalence classes $[\lambda : \mathcal{L} \to \mathbb{C}P^1]$ of N-fold branched coverings $\lambda : \mathcal{L} \to \mathbb{C}P^1$, where $\mathcal{L}$ is a compact Riemann surface of genus g and the holomorphic map λ of degree N is subject to the following conditions:

(i) it has M simple ramification points $P_1, \ldots, P_M \in \mathcal{L}$ with distinct *finite* images $\lambda_1, \ldots, \lambda_M \in \mathbb{C} \subset \mathbb{C}P^1$;
(ii) the preimage $\lambda^{-1}(\infty)$ consists of l points: $\lambda^{-1}(\infty) = \{\infty_1, \ldots, \infty_l\}$, and the ramification index of the map λ at the point ∞_j is k_j $(1 \leqslant k_j \leqslant N)$.

(We define the ramification index at a point as the number of sheets of the covering which are glued together at this point. A point ∞_j is a ramification point if and only if $k_j > 1$. A ramification point is simple if the corresponding ramification index equals 2.) The Riemann–Hurwitz formula implies that the dimension of this space is $M = 2g - 2 + l + N$.

Two branched coverings $\lambda : \mathcal{L}_1 \to \mathbb{P}^1$ and $\tilde{\lambda} : \mathcal{L}_2 \to \mathbb{P}^1$ are said to be equivalent if there exists a biholomorphic map $f : \mathcal{L}_1 \to \mathcal{L}_2$ such that $\tilde{\lambda} \circ f = \lambda$.

The meromorphic function λ on $\mathcal{L}$ defines a system of local parameters on the surface $\mathcal{L}$. Namely, $\lambda(P)$ can be used as a local coordinate everywhere except neighbourhoods of the ramification points $P_1, \ldots, P_M$ and the infinities $\infty_1, \ldots, \infty_l$. In a neighbourhood of the ramification point P_m the local parameter $x_m(P)$ is defined by the formula $x_m(P) = \sqrt{\lambda(P) - \lambda_m}$ and in the neighbourhood of the point ∞_s the local parameter $\zeta_s(P)$ is defined by the formula $\zeta_s(P) = (\lambda(P))^{-1/k_s}$.

Definition 1. *The local parameters x_m and ζ_s are called distinguished.*

We also introduce the covering $\widehat{H}_{g,N}(k_1, \ldots, k_l)$ of the space $H_{g,N}(k_1, \ldots, k_l)$ consisting of pairs

$$\left\langle (\mathcal{L}, \lambda) \in H_{g,N}(k_1, \ldots, k_l), \{a_\alpha, b_\alpha\}_{\alpha=1}^g \right\rangle,$$

where $\{a_\alpha, b_\alpha\}_{\alpha=1}^g$ is a canonical basis of cycles on the Riemann surface $\mathcal{L}$.

The spaces $\widehat{H}_{g,N}(k_1, \ldots, k_l)$ and $H_{g,N}(k_1, \ldots, k_l)$ are connected complex manifolds and the local coordinates on these manifolds are given by the images $\lambda_1, \ldots, \lambda_M$ of the ramification points under the map λ (or, what is the same, by the critical values of the meromorphic function λ).

For $g = 0$ the spaces $\widehat{H}_{g,N}(k_1, \ldots, k_l)$ and $H_{g,N}(k_1, \ldots, k_l)$ coincide.

2.2. Bergman tau-function

2.2.1. Definition of Bergman tau-function. Let λ be the meromorphic function on a Torelli marked Riemann surface $\mathcal{L}$ such that the pair $(\mathcal{L}, \lambda)$ represents an element of $\widehat{H}_{g,N}(k_1, \ldots, k_l)$. Let s_m, $m = 1, \ldots, M$, be the cycles $\{|x_m| = \epsilon\}$ with positive orientation encircling the zeros P_m of the differential $\mathrm{d}\lambda$. Here x_m is the distinguished local parameter near P_m and ϵ is a sufficiently small positive number (the discs $\{|x_m| \leqslant \epsilon\}$ should not intersect).

Let $S_{\mathrm{d}\lambda}$ be the projective connection defined by (1.3) with $\omega = \mathrm{d}\lambda$ then $S_B - S_{\mathrm{d}\lambda}$ is a quadratic differential and the ratio $(S_B - S_{\mathrm{d}\lambda})/\mathrm{d}\lambda$ is a meromorphic one-differential.

Define *Hamiltonians* H_m via the equations

$$H_m = -\frac{1}{12\pi \mathrm{i}} \oint_{s_m} \frac{S_B - S_{\mathrm{d}\lambda}}{\mathrm{d}\lambda}, \qquad m = 1, \ldots, M. \tag{2.1}$$

The Hamiltonians H_m are complex-valued functions on the Hurwitz space $\widehat{H}_{g,N}(k_1, \ldots, k_l)$. The following lemma was proved in [15].

Lemma 1. *Let $\lambda_1, \ldots, \lambda_M$ be the local coordinates on $\widehat{H}_{g,N}(k_1, \ldots, k_l)$ given by the critical values of the meromorphic function λ. Then one has the relation*

$$\frac{\partial H_m}{\partial \lambda_n} = \frac{\partial H_n}{\partial \lambda_m}, \qquad n, m = 1, \ldots, M. \tag{2.2}$$

Lemma 1 provides the flatness of the connection $d_B = d + \sum_{m=1}^{M} H_m \,\mathrm{d}\lambda_m$ in the trivial line bundle over $\widehat{H}_{g,N}(k_1, \ldots, k_l)$ and justifies the following definition.

Definition 2. *Let $\mathcal{U}$ be an open simply connected neighbourhood of a point $(\mathcal{L}, \lambda) \in \widehat{H}_{g,N}(k_1, \ldots, k_l)$, where $\lambda_1, \ldots, \lambda_M$ can be used as local coordinates. A holomorphic solution $\tau : \mathcal{U} \to \mathbb{C}$ of the system of equations*

$$\frac{\partial \log \tau}{\partial \lambda_m} = H_m, \qquad m = 1, \ldots, M, \tag{2.3}$$

is called the Bergman tau-function in $\mathcal{U}$.

Remark 1. Globally, the Bergman tau-function is defined as a horizontal section of some flat line bundle over $\widehat{H}_{g,N}(k_1, \ldots, k_l)$ constructed via the flat connection d_B. We refer the reader to [15] for further details.

2.2.2. Calculation of Bergman tau-function. Recall that the meromorphic differential $\mathrm{d}\lambda$ has M simple zeros P_m, $m = 1, \ldots, M$, and l poles of orders $k_1 + 1, \ldots, k_l + 1$ respectively. Let

$$\mathcal{D} = \sum_{n=1}^{M+l} d_n D_n$$

be the divisor of the meromorphic differential $\mathrm{d}\lambda$. Here $D_n = P_n$ and $d_n = 1$ for $n = 1, \ldots, M$, whereas for $s = 1, \ldots, l$ the point D_{M+s} is the sth pole of $\mathrm{d}\lambda$ and $d_{M+s} = -k_s - 1$. Due to the well-known property of the canonical divisor one has the relation

$$\mathcal{A}_P(\mathcal{D}) + 2K^P = \mathbf{m} + \mathbf{B}\mathbf{n}, \tag{2.4}$$

where $\mathbf{m}, \mathbf{n} \in \mathbb{Z}^g$ and $\mathcal{A}_P$ is the Abel map starting at point P. Moreover one can show that the fundamental polygon $\widehat{\mathcal{L}}$ can be chosen in such a way that the integer vectors $\mathbf{m}$ and $\mathbf{n}$ vanish.

Below we make the following agreement. If T is a differential of a real weight r and D_k is a point of divisor $\mathcal{D}$ then the expression $T(D_k)$ always denotes the value of T at the point D_k calculated in the corresponding distinguished local parameter (see definition 1). For example, one has $T(P_m) := T(x_m)(\mathrm{d}x_m)^{-r}|_{x_m=0}$ for $m = 1, \ldots, M$. Recall also that the prime form $E(P, Q)$ on the Riemann surface $\mathcal{L}$ has tensor weight $-1/2$ with respect to each of its arguments.

The following theorem gives an explicit local expression for the Bergman tau-function on Hurwitz space $\widehat{H}_{g,N}(k_1, \ldots, k_l)$ with $g > 0$. Its proof can be found in [17]. The analogue of this theorem in case $g = 0$ is much more elementary, it can be found in [15] or [19].

Theorem 1.

(i) Let $(\mathcal{L}, \lambda) \in \widehat{H}_{g,N}(k_1, \ldots, k_l)$ *with* $g > 0$ *and let*

$$\mathcal{F} = [\mathrm{d}\lambda(P)]^{\frac{g-1}{2}} \left\{ \prod_{k=1}^{M+l} E(P, D_k)]^{\frac{(1-g)d_k}{2}} \right\} \mathcal{C}(P), \tag{2.5}$$

where the differential $\mathcal{C}(P)$ *is defined in (1.5) and the fundamental polygon* $\widehat{\mathcal{L}}$ *is chosen in such a way that* $\mathcal{A}_P(\mathcal{D}) + 2K^P = 0$. *Then the expression* $\mathcal{F}$ *has tensor weight 0 with respect to P, is single-valued on* $\mathcal{L}$ *and, therefore, is P-independent.*

(ii) The Bergman tau-function τ *can be represented as follows:*

$$\tau = \mathcal{F}^{2/3} \prod_{r,s=1 r<s}^{M+l} [E(D_r, D_s)]^{\frac{d_r d_s}{6}}. \tag{2.6}$$

2.3. Bergman tau-function and Hermitian two-matrix model

The first application of the Bergman tau-function τ is connected with the theory of random matrices. It turned out that the genus one free energy of the Hermitian two-matrix model essentially coincides with $\frac{1}{2}\log\tau$. Here following [8] we give a brief outline of this result.

Consider the partition function of the Hermitian two-matrix model

$$\mathrm{e}^{-N^2 F} := \int \mathrm{d}M_1\, \mathrm{d}M_2\, \mathrm{e}^{-N\mathrm{tr}\{V_1(M_1)+V_2(M_2)-M_1M_2\}}, \tag{2.7}$$

where the integration goes over all independent matrix entries of $N \times N$ Hermitian matrices M_1 and M_2; V_1 and V_2 are two polynomial potentials (sometimes it is convenient to consider V_1 and V_2 as infinite power series). The expansion $F = \sum_{G=0}^{\infty} N^{-2G} F^G$ as $N \to \infty$ (so-called genus expansion) plays an important role in the theory, since the coefficients F^G appear both in statistical physics (Ising model) and in enumeration of genus G graphs (see, for example, [7]). If the polynomials V_1 and V_2 are of even degree with positive leading coefficients, then asymptotically, as $N \to \infty$, the main contribution to the partition function (2.7) is given by the matrices whose eigenvalues are concentrated in a finite set of intervals. The intervals filled by the eigenvalues of the matrix M_1 lie around the minima of the potential V_1; the eigenvalues of the matrix M_2 fill the intervals around the minima of the potential V_2.

The intervals supporting eigenvalues of matrices M_1 and M_2 correspond to the so-called spectral algebraic curve $\mathcal{L}$, defined by the equation

$$(V_1'(x) - y)(V_2'(y) - x) - \mathcal{P}^0(x, y) + 1 = 0, \tag{2.8}$$

where the polynomial of two variables $\mathcal{P}^0(x, y)$ is the zeroth-order term in $1/N^2$ expansion of the polynomial

$$\mathcal{P}(x, y) := \frac{1}{N} \left\langle \mathrm{tr} \frac{V_1'(x) - V_1'(M_1)}{x - M_1} \frac{V_2'(y) - V_2'(M_2)}{y - M_2} \right\rangle \tag{2.9}$$

(the notation $\langle Q(M_1, M_2)\rangle$ is used to define the expectation value of any functional Q of the matrices M_1 and M_2 with respect to the integration measure in (2.7)). The branch cuts of the spectral curve $\mathcal{L}$ corresponding to the projection of $\mathcal{L}$ on the x-plane coincide with the intervals supporting the eigenvalues of M_1 in the limit $N \to \infty$; the branch cuts corresponding to the projection of $\mathcal{L}$ on the y-plane are the intervals supporting the eigenvalues of M_2. If the convergency of the matrix integral (2.7) is not required (and (2.7) is understood as a formal power series in coefficients of the polynomials $V_{1,2}$), the genus of the spectral curve equals $d_1 d_2 - 1$ (so-called maximal genus).

The function $x : \mathcal{L} \to \mathbb{C}P^1$ has two poles, one of them is simple and another has order d_1, so the pair $(\mathcal{L}, x)$ belongs to the Hurwitz space $H_{d_1 d_2 - 1, d_1+1}(1, d_1)$. Let τ be the Bergman tau-function on $\widehat{H}_{d_1 d_2 - 1, d_1+1}(1, d_1)$. Suppose also that the potential V_2 is written in the form $V_2(y) = \sum_{k=1}^{d_2+1} \frac{v_k}{k} y^k$.

The following theorem was proved in [8].

Theorem 2. *The genus one contribution F^1 to the free energy F is given by the formula*

$$F^1 = \frac{1}{2}\log\tau + \frac{1}{24}\log\left\{(v_{d_2+1})^{1-\frac{1}{d_2}} \prod_{m=1}^{M} \operatorname{res}\bigg|_{P_m} \frac{(\mathrm{d}y)^2}{\mathrm{d}x}\right\} + C \tag{2.10}$$

where $P_1, \ldots, P_M$ are zeros of the differential $\mathrm{d}x$ on the spectral curve (i.e. the branch points of the spectral curve realized as a covering of the x-plane), which are assumed to be simple.

2.4. Bergman tau-function as isomonodromic tau-function of Frobenius manifold

Here we assume that the reader is familiar with the theory of Frobenius manifolds, in particular with constructions from chapter V of Dubrovin's lecture notes [4].

Let ϕ be a primary differential (see [4]) on the Riemann surface $\mathcal{L}$. This differential defines the structure of a semi-simple Frobenius manifold M_ϕ on $\widehat{H}_{g,N}(k_1, \ldots, k_l)$, including the following ingredients: the multiplication law on the tangent bundle $\partial_{\lambda_m} \circ \partial_{\lambda_n} = \delta_{mn}\partial_{\lambda_m}$; the unity $e = \sum_{m=1}^M \partial_{\lambda_m}$; the Euler field $E = \sum_{m=1}^M \lambda_m \partial_{\lambda_m}$ and the 1-form $\Omega_{\phi^2} = \sum_{m=1}^M \{\mathrm{Res}_{P_m}(\phi^2/\mathrm{d}\lambda)\}\,\mathrm{d}\lambda_m$.

The invariant metric $\xi(v, w) = \Omega_{\phi^2}(v \circ w)$ on the Frobenius manifold is flat and potential. In the coordinates $\lambda_1, \ldots, \lambda_M$ this metric is diagonal: $\xi = \sum_{m=1}^M \xi_{mm}(\mathrm{d}\lambda_m)^2$, $\xi_{mm} = \mathrm{Res}_{P_m}(\phi^2/\mathrm{d}\lambda)$. The rotation coefficients, $\gamma_{mn} (m \neq n)$, of this metric are defined by the equality

$$\gamma_{mn} = \frac{\partial_{\lambda_n}\sqrt{\xi_{mm}}}{\sqrt{\xi_{nn}}}.$$

The rotation coefficients γ_{mn} are independent of the primary differential ϕ, i.e. are the same for all Frobenius structures on the Hurwitz space $\widehat{H}_{g,N}(k_1, \ldots, k_l)$.

Let $\Gamma := \|\gamma_{mn}\|_{m,n=1,\ldots,M; m\neq n}, \mathcal{U} := \mathrm{diag}(\lambda_1, \ldots, \lambda_M)$ and $V := [\Gamma, \mathcal{U}]$. The isomonodromic tau-function τ_I of the semisimple Frobenius manifold M_ϕ is defined by the system of (compatible) equations

$$\frac{\partial \log\tau_I}{\partial\lambda_m} = H_m, \qquad m = 1, \ldots, M, \tag{2.11}$$

where the *quadratic Hamiltonians* H_m are defined by

$$H_m = \sum_{n\neq m; 1\leqslant n\leqslant M} \frac{V_{nm}^2}{\lambda_n - \lambda_m}, \qquad m = 1, \ldots, M. \tag{2.12}$$

Remark 2. Introduce matrices $A_k, k = 1, \ldots, M$, by $A_k = \|a^k_{mn}\|_{m,n=1,\ldots M}$, where $a^k_{mn} = 0$ if $m \neq k$ and $a^k_{kn} = V_{kn}$ for $n = 1, \ldots, M$. Moving the point $(\mathcal{L}, \lambda)$ in the Hurwitz space $\widehat{H}_{g,N}(k_1, \ldots, k_l)$, one gets an isomonodromic deformation of the Fuchsian system of differential equations

$$\frac{\mathrm{d}\Psi}{\mathrm{d}\lambda} + \sum_{k=1}^{M} \frac{A_k(\lambda_1, \ldots, \lambda_M)}{\lambda - \lambda_k}\Psi = 0. \tag{2.13}$$

The corresponding Jimbo–Miwa isomonodromic tau-function coincides with τ_I.

The following theorem was proved in [16].

Theorem 3. *The quadratic Hamiltonians (2.12) coincide with Hamiltonians (2.1). The isomonodromic tau-function τ_I of the semisimple Frobenius manifold M_ϕ coincides with the Bergman tau-function on Hurwitz space $\widehat{H}_{g,N}(k_1, \ldots, k_l)$.*

2.5. *Bergman tau-function and determinant of Laplacian*

Let $\det \Delta^{\mathbf{P}}$ be the ζ-regularized determinant of the Laplacian $\Delta^{\mathbf{P}}$ corresponding to the conformal metric of constant negative curvature -1 (the Poincaré metric) on $\mathcal{L}$. The Belavin–Knizhnik theorem implies that the second holomorphic–antiholomorphic derivatives (with respect to Teichmüller moduli) of the expression $\log Q^{\mathbf{P}}$, where

$$Q^{\mathbf{P}} = \frac{\det \Delta^{\mathbf{P}}}{\{\det \Im \mathbf{B}\}\{\mathrm{Area}(\mathcal{L})\}},$$

are nontrivial. In particular, Q fails to be the modulus square of a holomorphic function of moduli. Physicists (see, e.g., [14]) were first to propose the following factorization of the $\det \Delta^{\mathbf{P}}$,

$$\det \Delta^{\mathbf{P}} = \{\det \Im \mathbf{B}\}\, \mathrm{e}^{S} |F|^2, \tag{2.14}$$

where S is some (in fact, divergent and requiring a proper regularization) integral of the Dirichlet type, and F is a holomorphic function of moduli (the factor $\mathrm{Area}(\mathcal{L})$ is a moduli independent constant and can be omitted).

There are two known approaches to justify the factorization (2.14). The first one (see [22, 30, 31]) uses the Schottky uniformization of the surface $\mathcal{L}$. In the corresponding version of the factorization formula (2.14) F turns out to be a holomorphic function on Schottky space admitting explicit expression through generators of the Schottky group.

Another approach [15] uses the system of coordinates on $\mathcal{L}$ given by a meromorphic function on $\mathcal{L}$; this approach leads to a version of formula (2.14) with the Bergman tau-function playing the role of the factor F. To formulate this result we need some auxiliary constructions.

For $g > 1$ the Riemann surface $\mathcal{L}$ is biholomorphically equivalent to the quotient space $\mathbb{H}/\Gamma$, where $\mathbb{H} = \{z \in \mathbb{C} : \Im z > 0\}$; Γ is a strictly hyperbolic Fuchsian group. Denote by $\pi_F : \mathbb{H} \to \mathcal{L}$ the natural projection. Let x be a local parameter on $\mathcal{L}$. Introduce the standard metric of the constant curvature -1 on $\mathcal{L}$:

$$\mathrm{e}^{\chi} |\mathrm{d}x|^2 = \frac{|\,\mathrm{d}z|^2}{|\mathrm{Im}\, z|^2}, \tag{2.15}$$

where $z \in \mathbb{H}, \pi_F(z) = P, x = x(P)$. Take a point $(\mathcal{L}, \lambda)$ from the Hurwitz space $H_{g,N}(1, \ldots, 1)$. (For simplicity we are using meromorphic functions with simple poles.) Consider the distinguished system of local parameters on $\mathcal{L}$ corresponding to the function

λ. Introduce the real-valued functions $\chi(\lambda)$, $\chi^{\rm int}(x_m)$, $m = 1, \ldots, M$, and $\chi_n^\infty(\zeta_n)$, $n = 1, \ldots, N$, by specifying the local parameters $x = \lambda$, $x = x_m$ and $x = \zeta_n$ respectively. Consider domains $\mathcal{L}_\rho^{(n)}$ obtained from the nth sheet of the branched covering $\lambda : \mathcal{L} \to \mathbb{C}P^1$ by deleting small discs $\{|\lambda - \lambda_m| \leqslant \rho\}$ around the zeros of $d\lambda$ belonging to this sheet, and the disc $\{|\lambda| > 1/\rho\}$ around infinity. Introduce the following regularized Dirichlet integral:

$$\mathbb{D}_F := \frac{1}{\pi} \lim_{\rho \to 0} \left(\sum_{n=1}^N \int_{\mathcal{L}_\rho^{(n)}} |\partial_\lambda \chi|^2 \, \widehat{d\lambda} + (8N + M)\pi \log \rho \right). \tag{2.16}$$

Define also the function $\mathbb{S}_F$ by

$$\mathbb{S}_F(\lambda_1, \ldots, \lambda_M) = -\frac{1}{12}\mathbb{D}_F - \frac{1}{6} \sum_{m=1}^M \chi^{\rm int}(x_m) \Big|_{x_m=0} + \frac{1}{3} \sum_{n=1}^N \chi_n^\infty(\zeta_n) \Big|_{\zeta_n=0}. \tag{2.17}$$

The following theorem was proved in [15]:

Theorem 4. *Let a pair $(\mathcal{L}, \lambda)$ belong to the Hurwitz space $H_{g,N}(1, \ldots, 1)$. Then the determinant of the Laplace operator on $\mathcal{L}$ in the Poincaré metric is given by the following expression:*

$$\det \Delta^{\mathbf{P}} = c_{g,N} \{\det {\rm Im}\, \mathbf{B}\} \, e^{\mathbb{S}_F} |\tau|^2, \tag{2.18}$$

where $c_{g,N}$ is a constant independent of the point $(\mathcal{L}, \lambda) \in H_{g,N}(1, \ldots, 1)$; $\mathbf{B}$ is the matrix of b-periods on $\mathcal{L}$; τ is the Bergman tau-function on $H_{g,N}(1, \ldots, 1)$.

Formula (2.18) provides an alternative to existing ways of computation of $\det \Delta^{\mathbf{P}}$: via the Selberg zeta-function (see, for example, [5] and references therein), and via Zograf's F-function on Schottky space [31].

The presence of the holomorphic anomaly (i.e. the failure of $Q^{\mathbf{P}}$ to be the modulus square of a holomorphic function of coordinates on the Hurwitz space) can be related to the choice of Poincaré metric and one may hope that some other choice of a conformal metric will eliminate this anomaly.

The most radical choice is the metric $|d\lambda|^2$. This metric is flat and has conical singularities with conical angles 4π at the (simple) zeros of the differential $d\lambda$. It is known (see, e.g., [3]) that appearance of conical singularities of the metric does not change dramatically the spectral properties of the Laplacian, so the main problem will be related to the behaviour of this metric at other singular points, the poles of the meromorphic function λ (also assumed to be simple): due to the presence of these singularities the volume of $\mathcal{L}$ is infinite and the spectrum of the Laplacian is continuous. So even the definition of the determinant of this Laplacian presents a certain problem. We note that in a neighbourhood $\{|\lambda| > R\}$ of the point at infinity of each sheet of the covering $\lambda : \mathcal{L} \to \mathbb{C}P^1$ (i.e. a pole of $d\lambda$) the operator $\Delta^{|d\lambda|^2}$ coincides with the standard Laplacian in the exterior of the disc in the complex plane. This leads to a conjecture that the proper definition of $\det \Delta^{|d\lambda|^2}$ should resemble the one recently proposed in [11] for the determinants of Laplacians in exterior domains. This way to define the determinant can be briefly outlined as follows.

Let R be a sufficiently large positive number and let D_k, $k = 1, \ldots, N$, be the neighbourhoods $\{|\lambda| > R\}$ of the poles of $d\lambda$. Let Δ be (the closure of) the operator of the Dirichlet problem for the standard Laplacian $-4\partial_\lambda \partial_{\bar\lambda}$ in the exterior domain $\{\lambda \in \mathbb{C} : |\lambda| > R\}$. Let $\Delta_0 = \oplus_{k=1}^N \Delta \oplus \mathbf{0}$ be the operator acting as Δ in D_k and as 0 in $\mathcal{L} \setminus \cup_{k=1}^N D_k$. Then the operator $e^{-t\Delta^{|d\lambda|^2}} - e^{-t\Delta_0}$ for any $t > 0$ is of trace class and one can try to define the zeta-function $\zeta_{\Delta^{|\mathrm{d}\lambda|^2}}(s)$ of the operator $\Delta^{|\mathrm{d}\lambda|^2}$ via an appropriate regularization of the (diverging) integral

$$\frac{1}{\Gamma(s)}\int_0^\infty t^{s-1}\,\mathrm{tr}\big(\mathrm{e}^{-t\Delta^{|\mathrm{d}\lambda|^2}} - \mathrm{e}^{-t\Delta_0}\big)\,\mathrm{d}t. \tag{2.19}$$

Motivated by [11], we conjecture that this zeta-function has an expansion of the form $a_0 + a_1 s\log s + a_2 s + \cdots$ as $s \to 0-$ and setting

$$\det\Delta^{|\mathrm{d}\lambda|^2} = \mathrm{e}^{-a_2}$$

one gets a reasonable definition of the determinant of the Laplacian $\Delta^{|\mathrm{d}\lambda|^2}$ (up to an R-dependent factor).

The second part of our conjecture is that the Bergman tau-function and $\det\Delta^{|\mathrm{d}\lambda|^2}$ defined in this way are related as follows

$$\det\Delta^{|\mathrm{d}\lambda|^2} = C(R)\{\det\Im\mathbf{B}\}|\tau|^2,$$

where the constant $C(R)$ does not depend on the moduli (the coordinates on the Hurwitz space).

In the next section we shall realize in full a similar programme for flat metrics on $\mathcal{L}$ which are less singular than the metric $|\mathrm{d}\lambda|^2$. Namely, one can consider metrics given by $|W|$, where W is a meromorphic quadratic differential on $\mathcal{L}$ with at most simple poles. In this case the volume of $\mathcal{L}$ is finite, the corresponding Laplacian $\Delta^{|W|}$ has discrete spectrum and its determinant can be defined via standard ζ-regularization. The quantity

$$Q^{|W|} = \frac{\det\Delta^{|W|}}{\{\det\mathrm{Im}\,\mathbf{B}\}\{\mathrm{Area}(\mathcal{L}, |W|)\}}$$

coincides with the modulus square of a holomorphic function on the space of quadratic differentials (i.e. the moduli space of pairs $(\mathcal{L}, W)$)). This holomorphic function turns out to be a complete analogue of the Bergman tau-function on the Hurwitz space.

In particular, the holomorphic anomaly disappears if one takes instead of the conformal metric with uniformly distributed curvature the conformal metric $|W|$ whose curvature is supported at a finite number of points.

3. Bergman tau-function on spaces of quadratic differentials

3.1. Spaces of quadratic differentials with simple poles

The space $\mathcal{Q}_g$ of quadratic differentials on the Riemann surfaces of genus g is the moduli space of pairs $(\mathcal{L}, W)$, where $\mathcal{L}$ is a (Torelli marked) Riemann surface of genus g and W is a meromorphic quadratic differential on $\mathcal{L}$ having at most simple poles. The space $\mathcal{Q}_g$ is infinite-dimensional, since the number of poles can be arbitrary. This space is stratified as a family of finite-dimensional strata according to the multiplicities of zeros and the number of poles of quadratic differentials. Denote by $\mathcal{Q}_g(k_1, \ldots, k_{M_1}, l_1, \ldots, l_{M_2}, [-1]^L)$ the stratum of the space $\mathcal{Q}_g$ which consists of quadratic differentials having M_1 zeros of odd multiplicities $k_1, \ldots, k_{M_1}$, M_2 zeros of even multiplicities $l_1, \ldots, l_{M_2}$ and L simple poles, and which are not the squares of Abelian differentials (the last condition makes sense if $M_1 = L = 0$). Since the degree of divisor (W) equals $4g - 4$, the multiplicities of the zeros and the number of poles are connected by the equality $k_1 + \cdots + k_{M_1} + l_1 + \cdots + l_{M_2} - L = 4g - 4$. In particular, the number $L + M_1$ is always even. For any pair $(\mathcal{L}, W)$ from $\mathcal{Q}_g(k_1, \ldots, k_{M_1}, l_1, \ldots, l_{M_2}, [-1]^L)$ one can construct the so-called canonical two-fold covering

$$\kappa : \tilde{\mathcal{L}} \to \mathcal{L} \tag{3.1}$$

such that $\kappa^* W = w^2$, where w is a holomorphic 1-differential on $\tilde{\mathcal{L}}$. This covering is ramified over the poles and the zeros of odd multiplicity of W.

Counting the zeros of the holomorphic Abelian differential w on $\tilde{\mathcal{L}}$, we can compute the genus $\tilde{g}$ of the surface $\tilde{\mathcal{L}}$. Each zero of even multiplicity l_s of W gives rise to two distinct zeros of w of multiplicity $l_s/2$, whereas each zero of W of odd multiplicity k_s corresponds to a single zero of w of multiplicity $k_s + 1$. Thus, one has the relation

$$2\tilde{g} - 2 = l_1 + \cdots + l_{M_2} + k_1 + \cdots + k_{M_1} + M_1 = 4g - 4 + L + M_1, \tag{3.2}$$

therefore, $\tilde{g} = 2g + (L + M_1)/2 - 1$.

3.2. Coordinates on stratum $\mathcal{Q}_g(k_1, \ldots, k_{M_1}, l_1, \ldots, l_{M_2}, [-1]^L)$

The coordinates on the space $\mathcal{Q}_g(k_1, \ldots, k_{M_1}, l_1, \ldots, l_{M_2}, [-1]^L)$ can be constructed as follows ([20, 21, 23]). Let $R_1, \ldots, R_{M_1}$ and $P_1, \ldots, P_{M_2}$ be the zeros of a quadratic differential W of (respectively) odd and even multiplicities and let $S_1, \ldots, S_L$ be its poles.

Denote by $*$ the holomorphic involution on $\tilde{\mathcal{L}}$ interchanging the sheets of covering (3.1).

The differential $w(P)$ is anti-invariant with respect to involution $*$:

$$w(P^*) = -w(P). \tag{3.3}$$

Denote the canonical basis of cycles on $\mathcal{L}$ by (a_α, b_α). The canonical basis of cycles on $\tilde{\mathcal{L}}$ will be denoted as follows [9]:

$$\{a_\alpha, b_\alpha, a_{\alpha'}, b_{\alpha'}, \mathbf{a}_m, \mathbf{b}_m\}, \tag{3.4}$$

where $\alpha, \alpha' = 1, \ldots, g$; $m = 1, \ldots, (L + M_1)/2 - 1$; this basis can always be chosen to have the following invariance properties under the involution $*$:

$$a_\alpha^* + a_{\alpha'} = b_\alpha^* + b_{\alpha'} = 0 \tag{3.5}$$

and

$$\mathbf{a}_m^* + \mathbf{a}_m = \mathbf{b}_m^* + \mathbf{b}_m = 0. \tag{3.6}$$

For corresponding canonical basis of normalized holomorphic differentials $u_\alpha, u_{\alpha'}, \mathbf{u}_m$ on $\mathcal{L}$ we have as a corollary of (3.5), (3.6):

$$u_\alpha(P^*) = -u_{\alpha'}(P), \qquad \mathbf{u}_m(P^*) = -\mathbf{u}_m(P). \tag{3.7}$$

The canonical basis of normalized holomorphic differentials on $\mathcal{L}$ is then given by

$$w_\alpha(P) = u_\alpha(P) - u_{\alpha'}(P), \qquad \alpha = 1, \ldots, g. \tag{3.8}$$

The canonical meromorphic differential $\tilde{\mathbf{w}}(P, Q)$ on $\tilde{\mathcal{L}}$ satisfies the following relation:

$$\tilde{\mathbf{w}}(P^*, Q^*) = \tilde{\mathbf{w}}(P, Q) \tag{3.9}$$

for any $P, Q \in \tilde{\mathcal{L}}$; it is related to the meromorphic canonical differential $\mathbf{w}(P, Q)$ on $\mathcal{L}$ as follows:

$$\mathbf{w}(P, Q) = \tilde{\mathbf{w}}(P, Q) + \tilde{\mathbf{w}}(P, Q^*), \qquad P, Q \in \mathcal{L}. \tag{3.10}$$

Now we are to introduce the local coordinates on the stratum $\mathcal{Q}_g(k_1, \ldots, k_{M_1}, l_1, \ldots, l_{M_2}, [-1]^L)$. First consider the case when $L + M_1 > 0$ (quadratic differentials have a zero of odd multiplicity or a pole). The complex dimension of the stratum $\mathcal{Q}_g(k_1, \ldots, k_{M_1}, l_1, \ldots, l_{M_2}, [-1]^L)$ is $2g + (L + M_1 - 2) + M_2$. The first $2g + (L + M_1 - 2)$ coordinates on the space $\mathcal{Q}_g(k_1, \ldots, k_{M_1}, l_1, \ldots, l_{M_2}, [-1]^L)$ can be chosen by integrating the differential $w(P)$ over the basic cycles on $\tilde{\mathcal{L}}$ as follows ([20, 23]):

$$A_\alpha := \oint_{a_\alpha} w \qquad B_\alpha := \oint_{b_\alpha} w \qquad \mathcal{A}_m := \oint_{\mathbf{a}_m} w \qquad \mathcal{B}_m := \oint_{\mathbf{b}_m} w \tag{3.11}$$

for $\alpha = 1, \ldots, g, m = 1, \ldots, (L + M_1)/2 - 1$.

To introduce remaining M_2 coordinates we denote by P_k^+ and P_k^- the zeros of differential w such that $\pi\left(P_k^+\right) = \pi(P_k^-) = P_k$; $k = 1, \ldots, M_2$. The points $R_1, \ldots, R_{M_1}, S_1, \ldots, S_L$ have unique pre-images under the covering map κ (which are the ramification points of the covering). In the following, we shall denote these points and their pre-images by the same letters. Dissect the surface $\tilde{\mathcal{L}}$ along the basic cycles obtaining a fundamental polygon. Choose as a basic point one of the points $R_1, \ldots, R_{M_1}, S_1, \ldots, S_L$, say, R_1. Then the last M_2 coordinates are given by integrals of w over the paths connecting the basic point R_1 and the points $P_1^+, \ldots, P_{M_2}^+$:

$$z_k = \int_{R_1}^{P_k^+} w, \qquad k = 1, \ldots, M_2; \tag{3.12}$$

all the paths of integration lie inside the fundamental polygon.

Now consider the case $L + M_1 = 0$. In other words we deal here with holomorphic quadratic differentials which have only zeros of even multiplicity and are not the squares of Abelian differentials. (An example of such a differential can be found in [21].) In this case the canonical covering is unramified. The dimension of the stratum $\mathcal{Q}_g(l_1, \ldots, l_{M_2})$ is $2g+M_2-1$. The local coordinates are given by integrals

$$A_\alpha := \oint_{a_\alpha} w \qquad B_\alpha := \oint_{b_\alpha} w \qquad z_k := \int_{P_1^+}^{P_k^+} w, \tag{3.13}$$

where $\alpha = 1, \ldots g$; $k = 2, \ldots, M_2$.

In the following, we shall restrict ourselves to the case $M_1 + L > 0$. Treatment of the case $M_1 = L = 0$ is completely parallel to the treatment of the ramified covering case $M_1 + L > 0$.

If a quadratic differential W is the square of a holomorphic 1-differential w (in particular, all zeros of W have even multiplicity), then the canonical covering becomes a disjoint union of two copies $\mathcal{L}_+$ and $\mathcal{L}_-$ of the surface $\mathcal{L}$ and $\sqrt{\kappa^* W} = \pm w$ on $\mathcal{L}_\pm$. Thus, the moduli space of such quadratic differentials coincides with the moduli space of Abelian differentials.

3.3. *Bergman tau-function on spaces of quadratic differentials*

From now on we shall denote the coordinates $A_\alpha, B_\alpha, \mathcal{A}_m, \mathcal{B}_m, z_k$ on $\mathcal{Q}_g(k_1, \ldots, k_{M_1}, l_1, \ldots, l_{M_2}, [-1]^L)$ uniformly. Namely introduce the coordinates ζ_k, $k = 1, \ldots, 2g + (L + M_1 - 2) + M_2$ by

$$\zeta_k = \begin{cases} A_k, & \text{if} \quad k = 1, \ldots, g \\ B_{k-g}, & \text{if} \quad k = g+1, \ldots, 2g \\ \mathcal{A}_{k-2g}, & \text{if} \quad k = 2g+1, \ldots, 2g + (L+M_1)/2 - 1 \\ \mathcal{B}_{k-2g-(L+M_1)/2+1}, & \text{if} \quad k = 2g + (L+M_1)/2, \ldots, 2g + (L+M_1) - 2 \\ z_{k-2g-(L+M_1)+2}, & \text{if} \quad k = 2g + (L+M_1) - 1, \ldots, 2g + (L + M_1 - 2) + M_2. \end{cases}$$

Introduce also the elements s_k, $k = 1, \ldots, 2g+(L+M_1-2)+M_2$ of $H_1\left(\tilde{\mathcal{L}} \setminus \left\{P_1^+, \ldots, P_{M_2}^+\right\}, \mathbb{C}\right)$ by

$$s_k = \begin{cases} -b_k, & \text{if} \quad k = 1, \ldots, g \\ a_{k-g}, & \text{if} \quad k = g+1, \ldots, 2g \\ -\frac{1}{2}\mathbf{b}_{k-2g}, & \text{if} \quad k = 2g+1, \ldots, 2g + (L+M_1)/2 - 1 \\ \frac{1}{2}\mathbf{a}_{k-2g-(L+M_1)/2+1}, & \text{if} \quad k = 2g + (L+M_1)/2, \ldots, 2g + (L+M_1) - 2 \\ c_{k-2g-(L+M_1)+2}, & \text{if} \quad k = 2g + (L+M_1) - 1, \ldots, 2g + (L + M_1 - 2) + M_2, \end{cases}$$

where c_k is a small positively oriented contour encircling P_k^+.

Let S_B be the Bergman projective connection *on the base* $\mathcal{L}$ of the canonical covering. One can lift it to the covering surface $\tilde{\mathcal{L}}$ and we keep the same notation, S_B, for this lift (this is not the Bergman projective connection on $\tilde{\mathcal{L}}$!). We recall that $w = \sqrt{\kappa^* W}$ is an Abelian differential on $\tilde{\mathcal{L}}$. In complete analogy with constructions of section 2.2.1 introduce the Hamiltonians

$$\mathcal{H}_k = -\frac{1}{12\pi\mathrm{i}}\int_{s_k}\frac{S_B - S_w}{w}, \qquad k = 1, \ldots, 2g + (L + M_1 - 2) + M_2. \tag{3.14}$$

The following lemma was proved in [18]:

Lemma 2. *The relations hold*

$$\frac{\partial \mathcal{H}_k}{\partial \zeta_l} = \frac{\partial \mathcal{H}_l}{\partial \zeta_k}, \qquad k, l = 1, \ldots, 2g + (L + M_1 - 2) + M_2. \tag{3.15}$$

This lemma justifies the following definition.

Definition 3. *A holomorphic solution of the system of equations*

$$\frac{\partial \log \tau(\mathcal{L}, W)}{\partial \zeta_k} = -\frac{1}{12\pi\mathrm{i}}\oint_{s_k}\frac{S_B - S_w}{w}, \qquad k = 1, \ldots, 2g + (L + M_1 - 2) + M_2, \tag{3.16}$$

is called the Bergman tau-function $\tau(\mathcal{L}, W)$ *on the stratum* $\mathcal{Q}_g(k_1, \ldots, k_{M_1}, l_1, \ldots, l_{M_2}, [-1]^L)$ *of the space of quadratic differentials over the Riemann surface* $\mathcal{L}$.

As in the case of Hurwitz spaces system (3.16) can be integrated explicitly; to give the answer we need to introduce some auxiliary objects.

Definition 4 (cf [27])**.** *The local parameters on* $\mathcal{L}$

$$\lambda_s := \zeta_s^2 = \left(\int_{R_s}^P w\right)^{\frac{2}{k_s+2}}, \qquad \theta_r := \xi_r^2 = \left(\int_{S_r}^P w\right)^2, \qquad x_m(P) := \left(\int_{P_m}^P w\right)^{\frac{2}{l_m+2}} \tag{3.17}$$

near the points R_s, S_r *and* P_m *respectively, are called distinguished.*

Introduce the following notation for the divisor (W) of the quadratic differential W:

$$(W) = -\sum_{r=1}^{L} S_r + \sum_{s=1}^{M_1} k_s R_s + \sum_{m=1}^{M_2} l_m P_m := \sum_{k=1}^{L+M_1+M_2} n_k Q_k. \tag{3.18}$$

From now on we assume that this fundamental polygon is chosen in such a way that $\mathcal{A}_P((W)) + 4K^P = 0$, where $\mathcal{A}_P$ is the Abel map with the basepoint P. The following theorem was proved in [18]. As in case of Hurwitz spaces its analogue for $g = 0$ is much more elementary (cf section 3.5).

Theorem 5. *The Bergman tau-function on the space* $\mathcal{Q}_g(k_1, \ldots, k_{M_1}, l_1, \ldots, l_{M_2}, [-1]^L)$ *with* $g \geqslant 1$ *is given by the following expression:*

$$\tau(\mathcal{L}, W) = \mathcal{F}^{\frac{1}{3}} \prod_{k=1}^{L+M_1+M_2} \mathcal{C}^{\frac{n_k}{12(g-1)}}(Q_k), \tag{3.19}$$

where the expression

$$\mathcal{F} := W^{(g-1)/4}(P)\mathcal{C}(P)\prod_{k=1}^{L+M_1+M_2} E^{(1-g)n_k/4}(Q_k, P)$$

does not depend on $P \in \mathcal{L}$*; the prime-forms and differential* $\mathcal{C}$ *at the points of divisor (3.18) are evaluated in the distinguished local parameters given in definition 4.*

3.4. *Bergman tau-function on spaces of quadratic differentials and determinants of Laplacians in Strebel metrics*

Any meromorphic quadratic differential W with only simple poles defines a natural flat metric on the Riemann surface $\mathcal{L}$ given by $|W|$ (a Strebel metric). This metric has conical singularities at the zeros and poles of W. The cone angles of the metric $|W|$ equal π at the simple poles of W and $(k+2)\pi$ at the zeros of W of multiplicity k. The unbounded symmetric operator $4|W|^{-1}\partial\bar\partial$ in $L_2(\mathcal{L}, |w|^2)$ with domain $C_0^\infty(\mathcal{L}\backslash(W))$ admits the closure and has the self-adjoint Friedrichs extension which we denote by $\Delta^{|W|}$. It is known [3, 24] that the spectrum $\sigma(\Delta^{|W|})$ of $\Delta^{|W|}$ is discrete and

$$N(\lambda) = O(\lambda), \tag{3.20}$$

as $\lambda \to +\infty$, where $N(\lambda)$ is the number of eigenvalues of the positive operator $-\Delta^{|W|}$ not exceeding λ (counting multiplicity). The ζ-function of the operator $\Delta^{|W|}$ defined by the sum over positive eigenvalues:

$$\zeta(s) = \sum_{\lambda_j>0} \lambda_j^{-s}$$

for $\Re s > 1$ (in this domain the series converges due to asymptotics (3.20)) admits analytic continuation to a meromorphic function in $\mathbf{C}$ which is regular at $s = 0$ [12]. The regularized determinant of the operator $\Delta^{|W|}$ is defined by the equality

$$\det\Delta^{|W|} = \exp\{-\zeta'(0)\}.$$

The following theorem (proved in [18]) establishes the relation between the Bergman tau-function and the determinants of Laplacians in Strebel metrics.

Theorem 6. *Let a pair* $(\mathcal{L}, W)$ *belong to the space* $\mathcal{Q}_g(k_1,\dots,k_{M_1}, l_1,\dots,l_{M_2}[-1]^L)$. *The determinant of the Laplacian* $\Delta^{|W|}$ *admits the following expression through the Bergman tau-function on* $\mathcal{Q}_g(k_1,\dots,k_{M_1}, l_1,\dots,l_{M_2}[-1]^L)$*:*

$$\det\Delta^{|W|} = C\,\mathrm{Area}(\mathcal{L}, |W|)\{\det\Im\mathbf{B}\}|\tau(\mathcal{L}, W)|^2, \tag{3.21}$$

where C is a constant (which can be different for different connected components of this space). If $g = 0$

$$\det\Delta^{|W|} = C\,\mathrm{Area}(\mathcal{L}, |W|)|\tau(\mathcal{L}, W)|^2. \tag{3.22}$$

Determinants of Laplacians in flat metrics with conical singularities were studied in the physics literature via non-rigorous functional integral approach (see, for example, [6] and references therein); for metrics $|w|^2$ (w is an Abelian differential), when the cone angles are multiples of 2π, the formulae of [6] essentially coincide with ours under appropriate choice of local parameters at the conical points.

3.5. *Sphere with four conical singularities*

3.5.1. Bergman tau-function on $\mathcal{Q}_0([-1]^4)$. Here we illustrate the general framework in the case of the space $\mathcal{Q}_0([-1]^4)$.

This space can be considered as a space of equivalence classes of quadratic differentials on the Riemann sphere with four simple poles. Two such differentials W_1 and W_2 are called equivalent if there exists a Möbius transformation Θ such that $\Theta^*W_1 = W_2$. Take the quadratic differential

$$W = \frac{\mu_0(\mathrm{d}\lambda)^2}{(\lambda-\lambda_1)(\lambda-\lambda_2)(\lambda-\lambda_3)(\lambda-\lambda_4)}$$

with $\mu_0, \lambda_1, \ldots, \lambda_4 \in \mathbb{C}$ as a representative of a class $[W]$. Then the tau-function on $\mathcal{Q}_0([-1]^4)$ looks as follows:

$$\tau(W) = \mu_0^{-1/4} \prod_{i<j} (\lambda_i - \lambda_j)^{1/12}. \tag{3.23}$$

According to theorem 6, the determinant of the Laplacian on the Riemann sphere corresponding to the metric $|W|$ is given by the expression

$$\det \Delta^{|W|} = C \mathrm{Vol}(\mathbb{C}P^1, |W|) |\tau(\mathbb{C}P^1, W)|^2,$$

or, equivalently,

$$\det \Delta^{|W|} = C \prod_{i<j} |\lambda_i - \lambda_j|^{1/6} |\mu_0|^{1/2} \iint_{\mathbb{C}} \frac{|\mathrm{d}\lambda|^2}{|\lambda - \lambda_1||\lambda - \lambda_2||\lambda - \lambda_3||\lambda - \lambda_4|}. \tag{3.24}$$

The system (3.16) in the case of the space $\mathcal{Q}_0([-1]^4)$ contains only two equations:

$$\frac{\partial T(\mathcal{L}, |W|)}{\partial \mathcal{A}} = \frac{1}{24\pi \mathrm{i}} \oint_b \frac{S_B - S_w}{w}, \tag{3.25}$$

$$\frac{\partial T(\mathcal{L}, |W|)}{\partial \mathcal{B}} = -\frac{1}{24\pi \mathrm{i}} \oint_a \frac{S_B - S_w}{w}, \tag{3.26}$$

where

$$T(\mathcal{L}, |W|) = \log \frac{\det \Delta^W}{\mathrm{Vol}(\mathcal{L}, |W|)}.$$

Here a and b are a- and b-cycles on the elliptic surface $\tilde{\mathcal{L}}$; $\mathcal{A}$ and $\mathcal{B}$ are the corresponding periods of the 1-differential $w = \sqrt{\kappa^* W}$ and

$$2\mathrm{Vol}(\mathcal{L}, |W|) = \Im(\mathcal{A}\bar{\mathcal{B}}). \tag{3.27}$$

Moreover, one can integrate the system ((3.25) and (3.26)) directly. Let the elliptic surface $\tilde{\mathcal{L}}$ be given as a parallelogram in the complex plane with vertices $z_0, z_0 + \mathcal{A}, z_0 + \mathcal{A} + \mathcal{B}, z_0 + \mathcal{B}$ and with the opposite sides identified and let the differential w be $\mathrm{d}z$. One may assume that the centre of the parallelogram coincides with the origin and the projection $\pi : \tilde{\mathcal{L}} \to \mathcal{L}$ coincides with the factorization over the action of the holomorphic involution $z \mapsto -z$ (this involution has four fixed points.)

The canonical meromorphic bidifferential on $\tilde{\mathcal{L}}$ is given by

$$\tilde{\mathbf{w}}(z_1, z_2) = \left(\wp(z_1 - z_2) + \frac{2\eta_1}{\mathcal{A}} \right) \mathrm{d}z_1 \, \mathrm{d}z_2, \tag{3.28}$$

where $\wp$ is the Weierstrass function constructed from the lattice in the complex plane generated by $\mathcal{A}$ and $\mathcal{B}$, the constant $2\eta_1$ is the a-period of the Weierstrass ζ-function. Thus, the canonical bidifferential $\mathbf{w}$ on the base $\mathcal{L} = \mathbb{C}P^1$ in the local coordinate z takes the form

$$\mathbf{w}(z_1, z_2) = \tilde{\mathbf{w}}(z_1, z_2) + \tilde{\mathbf{w}}(z_1, -z_2),$$

which leads to the following expression for the projective connection S_B in the coordinate z

$$S_B(z) = -6\wp(2z). \tag{3.29}$$

Now the system (3.25)–(3.26) takes the form

$$\frac{\partial T(\mathcal{L}, |W|)}{\partial \mathcal{A}} = \frac{1}{2\pi \mathrm{i}} \eta_2, \tag{3.30}$$

$$\frac{\partial T(\mathcal{L}, |W|)}{\partial \mathcal{B}} = -\frac{1}{2\pi \mathrm{i}}\eta_1, \tag{3.31}$$

where $2\eta_2$ is the b-period of the Weierstrass ζ-function.

Let $\theta_k, k = 1, 2, 3, 4$ be the Jacobi theta-functions (see [29]). Introduce the Dedekind η-function $\eta(\sigma)$, (where $\sigma = \mathcal{B}/\mathcal{A}$), which is related to θ_1' as follows:

$$-2\pi\eta^3(\sigma) = \theta_1'(\sigma); \tag{3.32}$$

it can also be expressed via the product of even theta-constants using Jacobi identity

$$\theta_1'(\sigma) = \pi\theta_2(\sigma)\theta_3(\sigma)\theta_4(\sigma). \tag{3.33}$$

Using the Legendre identity $\mathcal{B}\eta_1 - \mathcal{A}\eta_2 = \pi \mathrm{i}$ together with the relation

$$\eta_1 = -\frac{1}{6\mathcal{A}}\frac{\theta_1'''}{\theta_1'}$$

and the heat equation for the Jacobi theta-function θ_1, we deduce from (3.30) and (3.31) the following expression for $T(\mathcal{L}, |W|)$:

$$T(\mathcal{L}, |W|) = \mathrm{const}\frac{|\eta(\mathcal{B}/\mathcal{A})|^2}{|\mathcal{A}|}. \tag{3.34}$$

This leads to the formula for the det $\Delta^{|W|}$:

$$\det \Delta^{|W|} = \mathrm{const}\frac{|\Im(\mathcal{A}\bar{\mathcal{B}})||\eta(\mathcal{B}/\mathcal{A})|^2}{|\mathcal{A}|}, \tag{3.35}$$

which is alternative to (3.24).

It is instructive to verify the coincidence of the right-hand sides of (3.24) and (3.35) by a direct calculation. Let $\mathbb{A}$ be the a-period of the holomorphic differential

$$\frac{\mathrm{d}\lambda}{\sqrt{(\lambda-\lambda_1)(\lambda-\lambda_2)(\lambda-\lambda_3)(\lambda-\lambda_4)}}.$$

Due to the Thomae formula, one has the following representation for the theta-constants

$$\theta_k^8 = \frac{1}{(2\pi)^4}\mathbb{A}^4(\lambda_{j_1}-\lambda_{j_2})^2(\lambda_{j_3}-\lambda_{j_4})^2,$$

where $k = 2, 3, 4$ and (j_1, j_2, j_3, j_4) are appropriate permutations of $(1, 2, 3, 4)$. This together with formula (3.32), the Jacobi identity (3.33) and the relation $\mathbb{A} = \mathcal{A}\sqrt{\mu_0}$ imply the equality

$$\eta(\mathcal{B}/\mathcal{A}) = C\mathcal{A}^{1/2}\mu_0^{-1/4}\prod_{i<j}(\lambda_i-\lambda_j)^{1/12}, \tag{3.36}$$

where C is a root of unity of the 24th degree. In turn, (3.36) implies the required relation

$$\frac{|\Im(\mathcal{A}\bar{\mathcal{B}})||\eta(\mathcal{B}/\mathcal{A})|^2}{|\mathcal{A}|} = \mathrm{const}\prod_{i<j}|\lambda_i-\lambda_j|^{1/6}|\mu_0|^{1/2}\iint_{\mathbb{C}}\frac{|\mathrm{d}\lambda|^2}{|\lambda-\lambda_1||\lambda-\lambda_2||\lambda-\lambda_3||\lambda-\lambda_4|}.$$

Remark 3. Formula (3.24) can also be deduced from formula (50) of the preprint [1].

3.5.2. An extremal problem. In [26] Sarnak conjectured that the function $-\log\det\Delta^{\mathbf{P}}$ is a Morse function on $\mathcal{M}_g$, the moduli space of compact Riemann surfaces of genus g. This fact, if true, can be used to study the topology of $\mathcal{M}_g$. The critical points of this function, the 'distinguished points' in the moduli space, should in particular include curves with large groups of symmetries (see [13] for discussion of genus two case). Similar questions can be asked about the Hurwitz spaces and spaces of Abelian and quadratic differentials. Here we discuss the simplest situation of the space $\mathcal{Q}_0([-1]^4)$.

Let $(\mathbb{C}P^1, W) \in \mathcal{Q}_0([-1]^4)$. Consider the extremal problem

$$\det\Delta^{|W|} \to \max, \qquad \text{when} \quad \mathrm{Vol}(\mathbb{C}P^1, |W|) = 1.$$

Imposing the condition $\mathrm{Vol}(\mathbb{C}\mathrm{P}^1, |\mathrm{W}|) = 1$, we get the following expression for the determinant $\det'\Delta^{|W|}$ of the Laplacian corresponding to the normalized metric $|W|$:

$$\det{}'\Delta^{|W|} = \mathrm{const}|\Im(\mathcal{A}/\mathcal{B})|^{1/2}|\eta(\mathcal{A}/\mathcal{B})|^2. \tag{3.37}$$

According to lemma 4.1 from [25], the $SL(2,\mathbb{Z})$-invariant function $f(\sigma) = |\Im\sigma|^{1/2}|\eta(\sigma)|^2$ achieves its global maximum at the point $\sigma = \mathrm{e}^{2\pi\mathrm{i}/3}$ of the fundamental domain $\mathcal{D} = \{\sigma : |\sigma| \geqslant 1, -1/2 \leqslant \Re\sigma < 1/2, \Im\sigma > 0\}$ of the modular group. Thus, we have proved the following proposition.

Proposition 1. *Let W be a meromorphic quadratic differential on the Riemann sphere with four simple poles. The determinant of the Laplacian corresponding to the normalized metric $|W|$ is maximal when*

$$\frac{\mathcal{B}}{\mathcal{A}} = \frac{p\,\mathrm{e}^{2\pi\mathrm{i}/3} + q}{r\,\mathrm{e}^{2\pi\mathrm{i}/3} + s},$$

where $p, q, r, s \in \mathbb{Z}$, $ps - qr = 1$.

Let us now rewrite this answer in terms of $\{\lambda_i\}$. Due to relation (3.24) the problem of maximization of $\det'\Delta^{|W|}$ reduces to the problem

$$\mathcal{F}(\lambda_1, \lambda_2, \lambda_3, \lambda_4) \to \max,$$

where

$$\mathcal{F}(\lambda_1, \lambda_2, \lambda_3, \lambda_4) = \prod_{i<j}|\lambda_i - \lambda_j|^{1/3}\iint_{\mathbb{C}}\frac{|\mathrm{d}\lambda|^2}{|\lambda-\lambda_1||\lambda-\lambda_2||\lambda-\lambda_3||\lambda-\lambda_4|}.$$

The function $\mathcal{F}$ remains invariant after application of the same Möbius transformation to all of its arguments, therefore,

$$\max\{\mathcal{F}; \mathbb{C}^4\} = \max\{\mathcal{F}(-1, 1, -k, k); k \in \mathbb{C}\},$$

where

$$k^2 = \frac{\theta_2^4(0|\sigma)}{\theta_3^4(0|\sigma)}$$

and $\sigma = \mathcal{B}/\mathcal{A}$. It is known (see, e.g., [2]) that k^2 takes four values $\mathrm{e}^{\pm\pi\mathrm{i}/3}, \mathrm{e}^{\pm 2\pi\mathrm{i}/3}$ when its argument runs through the set

$$\left\{\frac{p\,\mathrm{e}^{2\pi\mathrm{i}/3} + q}{r\,\mathrm{e}^{2\pi\mathrm{i}/3} + s} : p, q, r, s \in \mathbb{Z}, ps - qr = 1\right\}.$$

Thus, the function

$$\mathbb{C} \ni k \mapsto \mathcal{F}(-1, 1, -k, k)$$

achieves its absolute maximum at eight points $\pm\mathrm{e}^{\pi\mathrm{i}/6}, \pm\mathrm{e}^{-\pi\mathrm{i}/6}, \pm\mathrm{e}^{\pi\mathrm{i}/3}, \pm\mathrm{e}^{-\pi\mathrm{i}/3}$ of the unit circle.

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