A new hierarchy of integrable systems associated to Hurwitz spaces

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In this paper, we introduce a new class of integrable systems, naturally associated to Hurwitz spaces (spaces of meromorphic functions over Riemann surfaces). The critical values of the meromorphic functions play the role of 'times'. Our systems give a natural generalization of the Ernst equation; in genus zero, they realize the scheme of deformation of integrable systems proposed by Burtsev, Mikhailov and Zakharov. We show that any solution of these systems in rank 1 defines a flat diagonal metric (Darboux–Egoroff metric) together with a class of corresponding systems of hydrodynamic type and their solutions.

Keywords: deformations of integrable systems; Hurwitz spaces; Ernst equation;
systems of hydrodynamic type

1. Introduction

Deformations of Riemann surfaces play an important role in different branches of the theory of integrable systems. Implicit deformations of Riemann surfaces appear in the systems of hydrodynamic type and in Whitham deformations of integrable systems (Krichever 1989a,b; Dubrovin 1992). Explicit deformations of Riemann surfaces appear in algebro-geometric solutions of equations with variable spectral parameter (Korotkin 1989) and Frobenius manifolds (Dubrovin 1996, 1998).

The main representative of the class of systems with variable spectral parameter is the Ernst equation from general relativity, which has the form

$$\left((\xi - \bar{\xi})\, G_\xi G^{-1}\right)_{\bar{\xi}} + \left((\xi - \bar{\xi})\, G_{\bar{\xi}} G^{-1}\right)_\xi = 0,$$

where $G(\xi, \bar{\xi})$ is a matrix-valued function. The stationary axially symmetric Einstein equations are equivalent to this equation if G is a 2×2 matrix with some additional symmetries. The Ernst equation is the compatibility condition of the following linear system (Belinskii & Zakharov 1978; Maison 1978)

$$\frac{\partial \Psi}{\partial \xi} = \frac{G_\xi G^{-1}}{1 - \nu}\, \Psi, \quad \frac{\partial \Psi}{\partial \bar{\xi}} = \frac{G_{\bar{\xi}} G^{-1}}{1 + \nu}\, \Psi, \tag{1.1}$$

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where ν is the following function of spectral parameter $\lambda \in \mathbb{C}$ and variables $(\xi, \bar{\xi})$:

$$\nu = \frac{2}{\xi - \bar{\xi}} \left\{ \lambda - \frac{\xi + \bar{\xi}}{2} + \sqrt{(\lambda - \xi)(\lambda - \bar{\xi})} \right\}.$$

Function $\nu(\lambda)$ is nothing but the uniformization map of the genus zero Riemann surface which is the twofold covering of λ-plane with two branch points at $\lambda = \xi$ and $\lambda = \bar{\xi}$. The map $\lambda(\nu)$ is a rational map $\mathbb{C}P^1 \to \mathbb{C}P^1$ of degree two with critical values ξ and $\bar{\xi}$.

If the matrix dimension of G is 1, we can introduce the function $f = \ln G$ and the Ernst equation turns into the Euler–Darboux equation,

$$\frac{\partial^2 f}{\partial \xi \partial \bar{\xi}} + \frac{1}{2(\bar{\xi} - \xi)} \frac{\partial f}{\partial \xi} + \frac{1}{2(\xi - \bar{\xi})} \frac{\partial f}{\partial \bar{\xi}} = 0. \tag{1.2}$$

The natural question is whether the Ernst equation is an isolated example of an integrable system related to the space of rational maps or it is possible to define natural analogues of the Ernst equation which would correspond to spaces of rational maps of arbitrary degree. More general, is it possible to go beyond the spaces of rational maps, and define natural analogues of the Ernst equation corresponding to general Hurwitz spaces $H_{g,N}$ of meromorphic functions of degree N on Riemann surfaces of genus g? Is there any link between these higher analogues of Ernst equation and existing theories of systems of hydrodynamic type (Krichever 1989a,b; Dubrovin 1992), Frobenius manifolds and Darboux–Egoroff metrics corresponding to Hurwitz spaces (Dubrovin 1996)?

If we assume that ν in (1.1) is a constant, then the compatibility conditions of (1.1) leads to the equation of the principal chiral model; therefore, according to terminology proposed by Burtsev *et al.* (1987), it is natural to call the Ernst equation the 'deformation' of the principal chiral model equation. Burtsev *et al.* (1987) studied the general problem of deformation of a given integrable system which has U–V pair, where matrices U and V are meromorphic functions of constant spectral parameter ν. If one allows ν to depend on space variables (x, y), then the zero curvature condition $U_y - V_x + [U, V] = 0$ implies a set of differential equations for 'variable spectral parameter' ν and poles of matrices U and V. However, in Burtsev *et al.* (1987), no regular method was given to solve these differential equations.

The first goal of this paper is to fill this gap. Namely, we show that the deformation scheme of Burtsev *et al.* (1987) can be realized in terms of the spaces of rational maps $H_{0,N}$ of any given degree N. In this way, we get a new hierarchy of non-autonomous nonlinear integrable systems. We show how solutions of the new systems can be described in terms of matrix Riemann–Hilbert problem and isomonodromic deformations. If the matrix dimension equals 1, these nonlinear systems give rise to systems of linear non-autonomous second-order partial differential equations, generalizing the Euler–Darboux equation.

Second, we extend our framework to construct a class of new integrable systems starting from an arbitrary Hurwitz space $H_{g,N}$ (space of meromorphic functions of degree N on Riemann surfaces of genus g) for $g \geq 2$.

Third, in rank 1 (by rank of the integrable system we mean its matrix dimension) and any genus, we observe a very close relationship between our systems, Darboux–Egoroff metrics, systems of hydrodynamic type and Frobenius

manifolds. Moreover, our general formalism allows to give a simple description of systems of hydrodynamic type (as well as their solutions) associated to Hurwitz spaces $H_{g,N}$.

Let us describe our results in more details. Each rational map $R(\gamma)$ of degree N defines an N-fold branch covering $\mathcal{L}$ of $\mathbb{C}P^1$; a point P of $\mathcal{L}$ is a pair (λ, γ) such that

$$\lambda = R(\gamma). \tag{1.3}$$

We assume that $R(\gamma)$ maps the infinity in γ-plane to the infinity in λ-plane and $R(\gamma) = \lambda + o(1)$ as $\gamma \to \infty$; then function $R(\gamma)$ has the form

$$R(\gamma) = \gamma + \sum_{m=1}^{N-1} \frac{r_m}{\gamma - \mu_m}. \tag{1.4}$$

Denote the critical points of the map R by $\gamma_1, \ldots, \gamma_M$; we shall assume that all of them are simple; then $M = 2N - 2$ according to the Riemann–Hurwitz formula.

The ramification points of the covering $\mathcal{L}$ are denoted by $P_m = (\lambda_m, \gamma_m)$ (they are simple since all critical points of $R(\gamma)$ are simple); their projections λ_m on λ-plane are called the branch points of the covering $\mathcal{L}$ (we adopt the terminology of Fulton 1969); these are the values of the rational map $R(\gamma)$ at its critical points $\lambda_m = R(\gamma_m)$. In the sequel, we shall assume that all λ_m are different.

Introduce two functions on the covering $\mathcal{L}$: the function π, which projects $\mathcal{L}$ on λ-plane, $\pi(P) = \lambda$; and function ν, which projects $\mathcal{L}$ to γ-plane, $\nu(P) = \gamma$. The map $\nu : \mathcal{L} \to \mathbb{C}P^1$ is a one-to-one map; its inverse is nothing but a uniformization map of the covering $\mathcal{L}$. The maps ν and π are related as follows: $R(\nu(P)) = \pi(P)$.

Owing to our assumption about the behaviour of $R(\gamma)$ at infinity, in a neighbourhood of the infinite point on some (we shall call it the first) sheet of $\mathcal{L}$, the map $\nu(P)$ behaves as follows: $\nu(P) = \lambda + o(1)$.

The structure of Riemann surface on the branch covering $\mathcal{L}$ is defined as follows: in a neighbourhood of any point where $\mathcal{L}$ is non-ramified, we can consider λ as local parameter. In a neighbourhood of a ramification point P_m, the local coordinate is chosen to be $x_m = \sqrt{\lambda - \lambda_m}$.

The branch covering $\mathcal{L}$ is completely defined by the branch points λ_m and a representation of the fundamental group $\pi_1(\mathbb{C}P^1 \setminus \{\lambda_1, \ldots, \lambda_M\})$ in permutation group S_N. An element of the permutation group describes the permutation of sheets of the covering $\mathcal{L}$ if λ encircles a given contour in $\mathbb{C}P^1 \setminus \{\lambda_1, \ldots, \lambda_M\}$. We shall consider local deformations of the covering $\mathcal{L}$, such that this representation is kept fixed. Then the branch points λ_m can be considered as natural local coordinates on the space of rational maps; they will play the role of independent variables of our systems.

Let us fix some point P_0 of $\mathcal{L}$, such that its projection on the λ-plane $\lambda_0 = \pi(P_0)$ is independent of all $\{\lambda_m\}$; let $\gamma_0 \equiv \nu(P_0)$. Consider the following system of matrix linear differential equations for an auxiliary matrix-valued $r \times r$ function $\Psi(\lambda, \{\lambda_m\})$:

$$\left. \frac{\partial \Psi(P)}{\partial \lambda_m} \right|_{\pi(P)} = \frac{\gamma_0 - \gamma_m}{\nu(P) - \gamma_m} J_m \Psi, \tag{1.5}$$

where the derivatives are taken assuming that $\lambda = \pi(P)$ remains constant. We assume that the solution Ψ of (1.5) is normalized by the condition

$$\Psi(\lambda = \infty) = I. \tag{1.6}$$

If we put in (1.5) $P = P_0$, then the compatibility conditions of the system (1.5) imply that all J_m are logarithmic derivatives of a matrix-valued function $G(\{\lambda_m\})$: $J_m = G_{\lambda_m} G^{-1}$. Besides that, the compatibility conditions of the system (1.5) imply the following coupled system of non-autonomous nonlinear matrix partial differential equations of second order:

$$\left((\gamma_0 - \gamma_m) G_{\lambda_m} G^{-1} \right)_{\lambda_n} = \left((\gamma_0 - \gamma_n) G_{\lambda_n} G^{-1} \right)_{\lambda_m}, \tag{1.7}$$

for all $m, n = 1, \ldots, M$. We call the matrix dimension r of the matrix G the rank of the system (1.7).

For rational maps of degree two, the system (1.7) coincides with the ordinary Ernst equation (1.1). If the rank equals 1, the systems (1.7) give rise to certain generalization of the Euler–Darboux equation (1.2).

We define the tau-function of these systems by the following system of compatible equations

$$\frac{\partial}{\partial \lambda_m} \ln \tau = \frac{1}{2} \mathrm{res} \Big|_{P_m} \mathrm{tr} \left\{ \frac{(\mathrm{d}_P \Psi \Psi^{-1})^2}{\mathrm{d}\lambda} \right\}, \tag{1.8}$$

where $\mathrm{d}_P \Psi = (\partial \Psi / \partial \nu(P)) \mathrm{d}\nu(P)$; these equations fix τ up to an arbitrary constant multiplier if we assume that τ is a holomorphic function of $\{\lambda_m\}$.

Each system (1.7) possesses a subclass of 'isomonodromic' solutions which can be built from an arbitrary solution of the Schlesinger system in the same matrix dimension. For the isomonodromic sector, we relate the tau-function to the Jimbo–Miwa tau-function of the Schlesinger system. This link between two tau-functions generalizes the formula (found in Korotkin & Nicolai 1996) relating the so-called 'conformal factor' of the Ernst equation with the Jimbo–Miwa tau-function. We also show how to construct solutions of the systems (1.7) from solutions of matrix Riemann–Hilbert problems on $\mathbb{C}P^1$.

After developing the theory of the systems (1.7), associated to the spaces of rational maps $H_{0,N}$, we formulate a similar for Hurwitz spaces in arbitrary genus $H_{g,N}$.

Consider in detail the rank 1 case. Let the N-fold branched covering $\mathcal{L}$ have genus g; then the number of branch points is equal to $M = 2g + 2N - 2$ (as before, we assume that all the branch points are simple and have different projections on λ-plane). Again, the projections λ_m of the branch points P_m on λ-plane can be used as local coordinates on $H_{g,N}$. Let us introduce some basis of canonical cycles on $\mathcal{L}$. Denote by $E(P, Q)$ the prime form on $\mathcal{L}$, and by B_m the normalized (all a-periods vanish) Abelian differential of the second kind with the unique pole of second order at P_m with leading coefficient 1. Let us choose two arbitrary points P_0 and Q_0 on $\mathcal{L}$, such that their projections $\pi(P_0)$ and $\pi(Q_0)$ on the λ-plane are $\{\lambda_m\}$ independent. Then the $g > 0$ analogue of the genus zero systems (1.7) in rank 1 looks as follows:

$$\frac{\partial^2 f}{\partial \lambda_m \partial \lambda_n} - \frac{b(P_m, P_n)}{2} \left\{ \frac{v_n}{v_m} \frac{\partial f}{\partial \lambda_m} + \frac{v_m}{v_n} \frac{\partial f}{\partial \lambda_n} \right\} = 0, \quad m \neq n, \tag{1.9}$$

where

$$b(P_m, P_n) = \frac{B(P, Q)}{\mathrm{d}x_m(P)\mathrm{d}x_n(Q)} \bigg|_{P = P_m, Q = P_n},$$

and $B(P, Q) = \mathrm{d}_P \mathrm{d}_Q \ln E(P, Q)$ is the canonical meromorphic bidifferential; $v_m = \int_{Q_0}^{P_0} B_m$.

Solutions of the system (1.9) can be constructed as follows. Let l be an arbitrary closed contour on $\mathcal{L}$ such that its projection on the λ-plane $\pi(l)$ is independent of $\{\lambda_m\}$ and $P_m \notin l$ for any m. Let $h(P)$ be an arbitrary independent of $\{\lambda_m\}$ Hölder-continuous function on l. Then the function

$$f = \oint_l h(P) \mathrm{d}_P \ln \frac{E(P_0, P)}{E(Q_0, P)}, \qquad (1.10)$$

satisfies the system (1.9).

Any solution of the system (1.9) gives rise to a diagonal flat metric (Darboux–Egoroff metrics). Namely, consider the diagonal metric

$$\mathrm{d}s^2 = \sum_{m=1}^{M} g_{mm} \mathrm{d}\lambda_m^2, \qquad (1.11)$$

where $g_{mm} = \partial \ln \tau / \partial \lambda_m$ and $\tau(\{\lambda_m\})$ is the tau-function corresponding to an arbitrary solution of (1.9). The rotation coefficients of this metric turn out to be given by the canonical meromorphic bidifferential

$$\beta_{mn} = \frac{1}{2} b(P_m, P_n), \qquad (1.12)$$

i.e. they depend only on the covering $\mathcal{L}$ and do not depend on a particular solution of (1.9). In accordance with the Rauch variational formulae (Fay 1992), which describe dependence of the canonical meromorphic bidifferential on the branch points, these coefficients satisfy the equations

$$\frac{\partial \beta_{mn}}{\partial \lambda_l} = \beta_{ml} \beta_{ln},$$

for distinct l, m, n, which, together with annihilation of each β_{mn} by the operator $\sum_{k=1}^{M} \partial/\partial \lambda_k$, guarantee flatness of the metric (1.11).

The Darboux–Egoroff metrics are known to be closely related to integrable classes of systems of hydrodynamic type, including, in particular, Whitham equations for slow deformations of the Riemann surfaces in dispersionless limit of finite-gap solutions of integrable systems. These systems are solvable via so-called generalized hodograph method (Krichever 1989a,b; Tsarev 1990). Our present scheme gives a short and simple formulation of the theory of the systems of hydrodynamic type corresponding to Hurwitz spaces.

The paper is organized as follows. In §2, we derive an auxiliary system of differential equations which describe dependence of the critical points of the rational map of the form (1.4) on the critical values. Then we show integrability of the systems (1.7), define the corresponding tau-function and discuss the relationship of these systems to the Riemann–Hilbert problem and the Schlesinger system. Here, we also show how the systems (1.7) are related to deformations of integrable systems proposed by Burtzev *et al* (1987). In §3, we define analogues of the systems (1.7) related to Hurwitz spaces $H_{g,N}$ in arbitrary genus $g \geq 2$ and discuss their properties. In §4, we show that each solution of these systems in rank 1 defines a Darboux–Egoroff metric, and discuss related systems of hydrodynamic type in our framework, together with their solutions. Here, we also outline the link between higher genus analogues of systems (1.7) and isomonodromic deformations on algebraic curves. In §5, we discuss potential directions of future work.

2. Non-autonomous integrable systems related to spaces of rational maps

(a) Differential equations for critical points of rational maps

Consider a rational map $R(\gamma)$ of degree N of the form (1.4). If we choose the critical values $\lambda_1, \ldots, \lambda_M$ of the map (1.4) as independent parameters, each critical point γ_m becomes a function of all $\{\lambda_m\}$. The function $\nu(P)$ depends on the variables $\lambda_1, \ldots, \lambda_M$ as parameters. In the sequel, we shall denote the point of $\mathcal{L}$ which belongs to the jth sheet and has a projection λ on the λ-plane by $\lambda^{(j)}$. The map $\nu(P)$ has its only pole at the point at infinity of some sheet of $\mathcal{L}$; we enumerate the sheets of $\mathcal{L}$ in such a way that this sheet has number one; therefore,

$$\nu(P) = \lambda + o(1) \text{ as } P \to \infty^{(1)}. \tag{2.1}$$

Theorem 2.1. *Function $\nu(P)$, considered locally as function of λ and depending on branch points $\lambda_1, \ldots, \lambda_M$ as parameters, satisfies the following equations:*

$$\frac{\partial \nu}{\partial \lambda} = 1 + \sum_{n=1}^{M} \frac{\alpha_n}{\nu - \gamma_n}, \quad \frac{\partial \nu}{\partial \lambda_n} = -\frac{\alpha_n}{\nu - \gamma_n}, \tag{2.2}$$

where α_m are some functions of the branch points.

Proof. Consider the local behaviour of the function $\nu(P)$ near a branch point,

$$\nu(P) = \gamma_m + \kappa_m \sqrt{\lambda - \lambda_m} + O(\lambda - \lambda_m) \text{ as } P \to P_m. \tag{2.3}$$

From (2.3), we conclude that

$$\frac{\partial \nu}{\partial \lambda} = \frac{\kappa_m}{2\sqrt{\lambda - \lambda_m}} + O(1), \quad \frac{\partial \nu}{\partial \lambda_n} = -\delta_{mn} \frac{\kappa_m}{2\sqrt{\lambda - \lambda_m}} + O(1),$$

as $P \to P_m$. By (2.3), we can rewrite these expansions using $\nu(P)$ as the global coordinate on $\mathcal{L}$,

$$\frac{\partial \nu}{\partial \lambda} = \frac{\kappa_m^2}{2(\nu - \gamma_m)} + O(1), \quad \frac{\partial \nu}{\partial \lambda_n} = -\delta_{mn} \frac{\kappa_m^2}{2(\nu - \gamma_n)} + O(1), \tag{2.4}$$

where $\gamma_n = \nu(P_n)$. Moreover, from (2.1), we conclude that $\partial \nu / \partial \lambda = 1 + o(1)$ and $\partial \nu / \partial \lambda_n = o(1)$ as $\nu(P) \to \infty$. Therefore, $\partial \nu / \partial \lambda$ is a meromorphic function of ν with simple poles at all the points γ_n with residues $\kappa_n^2/2$ and value 1 at infinity. Analogously, the function $\partial \nu / \partial \lambda_n$ is a meromorphic function on $\mathbb{C}P^1$ with simple pole at λ_n and zero at infinity. Therefore, we get equations (2.2) with $\alpha_n = \kappa_n^2/2$. ∎

Corollary 2.1. *The critical points γ_m of the rational function (1.4) and residues α_m from (2.2) depend on the critical values λ_m as follows:*

$$\frac{\partial \gamma_m}{\partial \lambda_n} = \frac{\alpha_n}{\gamma_n - \gamma_m}, \quad m \neq n; \quad \frac{\partial \gamma_m}{\partial \lambda_m} = 1 + \sum_{n=1, n \neq m}^{M} \frac{\alpha_n}{\gamma_m - \gamma_n}, \tag{2.5}$$

$$\frac{\partial \alpha_m}{\partial \lambda_n} = \frac{2\alpha_n \alpha_m}{(\gamma_n - \gamma_m)^2}, \quad m \neq n; \quad \frac{\partial \alpha_m}{\partial \lambda_m} = -\sum_{n=1, n \neq m}^{M} \frac{2\alpha_n \alpha_m}{(\gamma_n - \gamma_m)^2}, \tag{2.6}$$

for all $m, n = 1, \ldots, M$.

Proof. Equations (2.5) and (2.6) follow from the compatibility of equations (2.2). ∎

Rational functions of the form (1.4) were introduced by Kupershmidt & Manin (1977) in connection with Benney systems. The fact that the critical points of these functions satisfy equations (2.5) and (2.6) follows from the recent paper by Gibbons & Tsarev (1999).[1] However, we did not find in existing literature the whole set of equations (2.2), (2.5), (2.6) associated to the functions (1.4).

(i) *Twofold coverings*

Consider the simplest case $N=2$, when the covering $\mathcal{L}$ has two sheets and two branch points λ_1, λ_2. Then the rational function $R(\gamma)$ (1.4) can be explicitly written in terms of its critical values λ_1 and λ_2,

$$R(\gamma) = \gamma + \frac{(\lambda_1 - \lambda_2)^2}{16(\gamma - (\lambda_1 + \lambda_2)/2)}. \tag{2.7}$$

The map $\nu(P)$ looks as follows:

$$\nu(P) = \frac{1}{2}\left(\lambda + \frac{\lambda_1 + \lambda_2}{2} + \sqrt{(\lambda - \lambda_1)(\lambda - \lambda_2)}\right). \tag{2.8}$$

The critical points $\gamma_{1,2}$ and variables $\alpha_{1,2}$ are given by

$$\gamma_1 \equiv \nu(\lambda_1) = \frac{3\lambda_1 + \lambda_2}{4}, \quad \gamma_2 \equiv \nu(\lambda_2) = \frac{\lambda_1 + 3\lambda_2}{4}, \tag{2.9}$$

$$\alpha_1 = -\alpha_2 = \frac{\lambda_1 - \lambda_2}{8}. \tag{2.10}$$

(*b*) *Spaces of rational maps and new hierarchy of non-autonomous integrable systems*

Starting from an arbitrary branch N-fold covering $\mathcal{L}$ of genus zero, we can construct a hierarchy of integrable systems as follows.

Fix some point $P_0 \in \mathcal{L}$ such that its projection λ_0 on $\mathbb{C}P^1$ is independent of all $\{\lambda_m\}$, i.e. $\gamma_0 := \nu(P_0)$ depends on $\{\lambda_m\}$ according to the equation

$$R(\gamma_0(\lambda_1, ..., \lambda_M)) = \lambda_0.$$

Consider the following system of first-order differential equations for a $r \times r$ matrix-valued function $\Psi(P, \{\lambda_m\})$:

$$\left.\frac{\partial \Psi}{\partial \lambda_m}\right|_{\pi(P)} = \frac{\gamma_0 - \gamma_m}{\nu(P) - \gamma_m} J_m \Psi, \tag{2.11}$$

where J_m are $r \times r$ matrix-valued functions of $\{\lambda_m\}$; variable $\lambda = \pi(P)$ remains fixed under differentiation.

As a corollary of compatibility conditions of the linear system (2.11), functions J_m can be expressed in terms of the single function $G \equiv \Psi(P_0)$,

$$J_m = \frac{\partial G}{\partial \lambda_m} G^{-1}; \tag{2.12}$$

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[1] We thank E. Ferapontov, who attracted our attention to this work.

moreover, the function G satisfies the following system of nonlinear partial differential equations:

$$\left((\gamma_0 - \gamma_m)\, G_{\lambda_m} G^{-1}\right)_{\lambda_n} = \left((\gamma_0 - \gamma_n)\, G_{\lambda_n} G^{-1}\right)_{\lambda_m}. \qquad (2.13)$$

(to derive (2.13) from (2.11) one needs to make use of equations (2.2), (2.5) and (2.6)).

Alternatively, consider $\Psi(P)$ as a function of $\nu(P)$ and $\lambda_1, \ldots, \lambda_M$. Then, since $\nu(P)$ is itself a function of λ and $\{\lambda_m\}$, we can rewrite (2.11) using the chain rule

$$\left.\frac{\partial \Psi}{\partial \lambda_m}\right|_{\pi(P)} = \left.\frac{\partial \Psi}{\partial \lambda_m}\right|_{\nu(P)} + \frac{\partial \nu}{\partial \lambda_m}\frac{\partial \Psi}{\partial \nu},$$

where the partial derivative $\partial \Psi / \partial \lambda_m$ in the r.h.s. is taken for fixed $\nu(P)$.

Then, using (2.2) for $\partial \nu / \partial \lambda_m$, we rewrite (2.11) as follows:

$$\left.\frac{\partial \Psi(P)}{\partial \lambda_m}\right|_{\nu(P)} - \frac{\alpha_m}{\nu(P) - \gamma_m}\frac{\partial \Psi(P)}{\partial \nu(P)} = \frac{\gamma_0 - \gamma_m}{\nu(P) - \gamma_m} J_m \Psi(P). \qquad (2.14)$$

A simple calculation making use of equations (2.13) and system (2.5) and (2.6) shows that if $G(\{\lambda_m\})$ is a solution of nonlinear system (2.13), then the 1-form

$$q = \sum_{m=1}^{M} \frac{(\gamma_0 - \gamma_m)^2}{2\alpha_m}\, \mathrm{tr}\left(G_{\lambda_m} G^{-1}\right)^2 \mathrm{d}\lambda_m, \qquad (2.15)$$

is closed, $\mathrm{d}q = 0$.

The closedness of the 1-form q implies the existence of the potential τ, which can naturally be called the tau-function of the nonlinear system (2.13).

Definition 2.1. The function $\tau(\lambda_1, \ldots, \lambda_M)$, defined by the following system of equations:

$$\frac{\partial \ln \tau}{\partial \lambda_m} = \frac{(\gamma_0 - \gamma_m)^2}{2\alpha_m}\, \mathrm{tr}\left(G_{\lambda_m} G^{-1}\right)^2, \qquad (2.16)$$

up to an arbitrary constant multiplier, is called the tau-function of integrable system (2.13).

Using equations (2.13), (2.5) and (2.6), we find that the second derivatives of the tau-function are given by the following expression:

$$\frac{\partial^2 \ln \tau}{\partial \lambda_m \partial \lambda_n} = \frac{(\gamma_0 - \gamma_m)(\gamma_0 - \gamma_n)}{2(\gamma_m - \gamma_n)^2}\, \mathrm{tr}\left\{G_{\lambda_m} G^{-1} G_{\lambda_n} G^{-1}\right\}, \qquad (2.17)$$

for $m \neq n$.

Taking the residue of the linear system (2.14) at $P = P_m$, we find

$$\alpha_m \Psi_\gamma \Psi^{-1}(P_m) = (\gamma_0 - \gamma_m)\, G_{\lambda_m} G^{-1}; \qquad (2.18)$$

due to this relation, the definition (2.16) allows an alternative formulation.

Definition 2.1 The tau-function of the system (2.13) is defined by the following equations:

$$\frac{\partial \ln \tau}{\partial \lambda_m} = \frac{1}{2}\, \mathrm{res}\bigg|_{P_m}\left\{\frac{\mathrm{tr}(\mathrm{d}_P \Psi \Psi^{-1})^2}{\mathrm{d}\lambda}\right\}, \qquad (2.19)$$

where

$$d_P \boldsymbol{\Psi} \boldsymbol{\Psi}^{-1} \equiv \boldsymbol{\Psi}_{\nu(P)} \boldsymbol{\Psi}^{-1} d\nu(P). \tag{2.20}$$

Let us prove the equivalence of the two definitions of the tau-function. Using $\nu(P)$ as a global coordinate on $\mathcal{L}$, we have

$$d\lambda = \frac{\partial \lambda}{\partial \nu} d\nu;$$

therefore,

$$\frac{\text{tr}\{(d_P \boldsymbol{\Psi} \boldsymbol{\Psi}^{-1})^2\}}{d\lambda} = \frac{\partial \nu}{\partial \lambda} \text{tr}\{(\boldsymbol{\Psi}_{\nu} \boldsymbol{\Psi}^{-1})^2\} d\nu = \left\{ 1 + \sum_{m=1}^{M} \frac{\alpha_m}{\nu - \gamma_m} \right\} \text{tr}\{(\boldsymbol{\Psi}_{\nu} \boldsymbol{\Psi}^{-1})^2\} d\nu,$$

and (2.19) coincides with (2.16).

(ii) *The Ernst equation*

For $N=2$, the hierarchy (2.13) reduces to a single equation. If one chooses point P_0 to coincide with $\infty^{(2)}$ (i.e. the point of $\mathcal{L}$ where $\lambda = \infty$ and in a neighbourhood of which $\sqrt{(\lambda - \lambda_1)(\lambda - \lambda_2)} = -\lambda + (\lambda_1 + \lambda_2)/2 + o(1)$), we get

$$\gamma_0 = \nu(P_0) = \frac{\lambda_1 + \lambda_2}{2}.$$

Taking into account expressions (2.9), we have

$$\gamma_0 - \gamma_1 = \frac{\lambda_2 - \lambda_1}{4}, \quad \gamma_0 - \gamma_2 = \frac{\lambda_1 - \lambda_2}{4}. \tag{2.21}$$

If we now assume that λ_1 and λ_2 are conjugated to each other, $\lambda_1 = \xi$, $\lambda_2 = \bar{\xi}$, then equation (2.13) takes the following form:

$$\left((\xi - \bar{\xi}) G_\xi G^{-1} \right)_{\bar{\xi}} + \left((\xi - \bar{\xi}) G_{\bar{\xi}} G^{-1} \right)_\xi = 0. \tag{2.22}$$

This equation is called the Ernst equation; it is equivalent to vacuum Einstein's equation for stationary axially symmetric spacetimes (the matrix G in this case must be real, symmetric and must have unit determinant). In this case, the linear system (2.11) is equivalent to the Lax representation of the Ernst equation found by Belinskii & Zakharov (1978) and Maison (1978). We note that the Maison's Lax pair is a partial case of our linear system written in the form (2.11), whereas the Belinskii–Zakharov linear system is a partial case of (2.14).

Owing to expressions (2.10) for $\alpha_{1,2}$, the definition of the tau-function τ can be written down as follows:

$$\frac{\partial \ln \tau}{\partial \xi} = \frac{\xi - \bar{\xi}}{4} \text{tr}\left(G_\xi G^{-1} \right)^2, \quad \frac{\partial \ln \tau}{\partial \bar{\xi}} = \frac{\bar{\xi} - \xi}{4} \text{tr}\left(G_{\bar{\xi}} G^{-1} \right)^2. \tag{2.23}$$

Formula (2.23) coincides with the definition of the so-called conformal factor—one of the metric coefficients which correspond to a given solution of the Ernst equation.

(*c*) *New integrable systems and deformation scheme of Burtzev–Mikhailov–Zakharov*

The possibility to construct the class of 'deformed' integrable systems, or integrable systems with variable spectral parameter, different from the Ernst equation, was first discovered by Burtzev *et al.* (1987). They proposed to consider the Lax pairs of the form

$$\frac{\partial \Psi}{\partial x} = U\Psi, \quad \frac{\partial \Psi}{\partial y} = V\Psi, \tag{2.24}$$

where x and y are independent variables; matrices U and V depend on (x, y) and the variable spectral parameter ν (which in turn depends on (x, y) and the 'hidden' spectral parameter λ),

$$U(x, y, \nu) = u_0(x, y) + \sum_{n=1}^{N_1} \frac{u_n(x, y)}{\nu(x, y) - \gamma_n(x, y)}, \tag{2.25}$$

$$V(x, y, \nu) = v_0(x, y) + \sum_{n=1}^{N_2} \frac{v_n(x, y)}{\nu(x, y) - \tilde{\gamma}_n(x, y)}. \tag{2.26}$$

As a part of compatibility conditions of the linear system (2.24), after an appropriate fractional linear transformation in the ν-plane, the following system of equations for $\nu(x, y, \lambda)$ must be satisfied

$$\frac{\partial \nu}{\partial y} + \sum_{m=1}^{N_2} \frac{b_m}{\nu - \tilde{\gamma}_m} = 0, \quad \frac{\partial \nu}{\partial x} + \sum_{m=1}^{N_1} \frac{c_m}{\nu - \gamma_m} = 0, \tag{2.27}$$

where b_n and c_n are certain functions of (x, y). The compatibility condition of the system (2.27) gives the following system for $\gamma_n(x, y)$ and $\tilde{\gamma}_n(x, y)$:

$$\frac{\partial \gamma_n}{\partial y} + \sum_{m=1}^{N_2} \frac{b_m}{\gamma_n - \tilde{\gamma}_m} = 0, \quad \frac{\partial \tilde{\gamma}_n}{\partial x} + \sum_{m=1}^{N_1} \frac{c_m}{\tilde{\gamma}_n - \gamma_m} = 0, \tag{2.28}$$

$$\frac{\partial c_n}{\partial y} - 2c_n \sum_{m=1}^{N_2} \frac{b_m}{(\gamma_n - \tilde{\gamma}_m)^2} = 0, \quad \frac{\partial b_n}{\partial x} - 2b_n \sum_{m=1}^{N_1} \frac{c_m}{(\tilde{\gamma}_n - \gamma_m)^2} = 0. \tag{2.29}$$

It is easy to establish the relationship between solutions of the system (2.27), (2.28) and (2.29), and solutions of our system (2.2), (2.5) and (2.6).

Namely, suppose that function $\nu(\lambda, \{\lambda_m\}_{m=1}^M)$ satisfies equations (2.2) with respect to variables λ_m. Assume that $M = N_1 + N_2$ and split the set of variables $\{\lambda_1, ..., \lambda_{N_1+N_2}\}$ into two subsets: $\{\lambda_1, ..., \lambda_{N_1}\}$ and $\{\tilde{\lambda}_1, ..., \tilde{\lambda}_{N_2}\}$, where $\tilde{\lambda}_n \equiv \lambda_{N_1+n}, n = 1, ..., N_2$. In the same way, we split the set $\{\gamma_m\}$ of values of function $\nu(P)$ at these points,

$$\{\gamma_1, ..., \gamma_M\} = \{\gamma_1, ..., \gamma_{N_1}\} \cup \{\tilde{\gamma}_1, ..., \tilde{\gamma}_{N_2}\},$$

where $\tilde{\gamma}_n \equiv \gamma_{N_1+n}, n = 1, ..., N_2$.

Now assume that the 'untilded' variables $\lambda_1, \ldots, \lambda_{N_1}$ are arbitrary functions of variable x and the 'tilded' variables $\tilde{\lambda}_1, \ldots, \tilde{\lambda}_{N_2}$ are arbitrary functions of variable y. Then using (2.2), we get the derivative of $\nu(P)$ with respect to x,

$$\frac{\partial \nu}{\partial x} = \sum_{m=1}^{N_1} \frac{\partial \nu}{\partial \lambda_m} \frac{\partial \lambda_m}{\partial x} = -\sum_{m=1}^{N_1} \frac{\partial \lambda_m}{\partial x} \frac{\alpha_m}{\nu - \gamma_m}; \tag{2.30}$$

therefore,

$$\frac{\partial \nu}{\partial x} + \sum_{m=1}^{N_1} \frac{c_m}{\nu - \gamma_m} = 0, \tag{2.31}$$

where $c_m \equiv \alpha_m \frac{\partial \lambda_m}{\partial x}$; this coincides with the second equation in (2.27). The first equation in (2.27) is obtained in the same way after identification

$$b_m \equiv \alpha_{N_1+m} \frac{\partial \lambda_{N_1+m}}{\partial y}.$$

Equations (2.28) and (2.29) for γ_n, b_n and c_n as functions of (x, y) arise as compatibility conditions of equations for ν_x and ν_y.

Therefore, spaces of rational maps of given degree provide solutions of the system (2.27), (2.28) and (2.29) if we

(i) split the set of the branch points $\{\lambda_m\}$ into two subsets and
(ii) assume that one subset contains the branch points which are arbitrary functions of x only and another subset contains the branch points which are arbitrary functions of y only.

Concluding, we see that the system (2.13) provides realizations of the deformation scheme of Burtzev, Mikhailov and Zakharov.

(d) Solutions via the Riemann–Hilbert problem

Solutions of systems (2.13) in terms of matrix Riemann–Hilbert problem are given by the following theorem.

Theorem 2.2. *Consider a closed contour l on the branch covering $\mathcal{L}$ such that its projection $\pi(l)$ on the λ-plane is independent of $\{\lambda_m\}$. Assume that none of the ramification points P_m belongs to l. Define a non-degenerate matrix function $l \ni P \mapsto H(P)$ which is independent of $\{\lambda_m\}$. Suppose that a function $\Psi(P)$ satisfies the following Riemann–Hilbert problem on $\mathcal{L}$:*

(i) *$\Psi(P)$ is holomorphic and non-degenerate on $\mathcal{L}$ outside of the contour l where it has the finite boundary values related as follows:*

$$\Psi_+(P) = \Psi_-(P)H(P). \tag{2.32}$$

(ii) *Ψ satisfies the normalization condition at $\infty^{(1)}$ (this is the point of $\mathcal{L}$ such that $\pi(\infty^{(1)}) = \infty$ and $\nu(\infty^{(1)}) = \infty$):*

$$\Psi(\infty^{(1)}) = I. \tag{2.33}$$

Then the function Ψ satisfies the linear system (2.11) with

$$G = \Psi(P_0), \tag{2.34}$$

and, therefore, the function G solves the system (2.13).

Proof. Let Ψ satisfy the Riemann–Hilbert problem. Consider its logarithmic derivative $\Psi_{\lambda_m}\Psi^{-1}|_{\pi(P)}$. Owing to the independence of the contour l and the matrix $H(P)$ of λ_m, this logarithmic derivative is single valued on $\mathcal{L}$. Moreover, it is obviously holomorphic on $\mathcal{L}$ outside of the point P_m. Let us write the first two terms of the Taylor expansion of $\Psi(P)$ near P_m

$$\Psi = \Psi_0 + \sqrt{\lambda - \lambda_m}\Psi_1 + \cdots;$$

then

$$\Psi_{\lambda_m}\Psi^{-1}\bigg|_\lambda = \frac{-1}{2\sqrt{\lambda - \lambda_m}}\Psi_1\Psi_0^{-1} + O(1).$$

Therefore, in terms of the uniformization map $\nu(P)$, we can write

$$\Psi_{\lambda_m}\Psi^{-1}\bigg|_{\pi(P)} = \frac{C_m}{\nu(P) - \gamma_m},$$

with some matrix C_m which is independent of P. The constant term in this formula is absent due to normalization condition (2.33). To compute C_m, we put $P = P_0$, i.e. $\nu(P)$; then using (2.34), we get

$$C_m = (\gamma_0 - \gamma_m)G_{\lambda_m}G^{-1},$$

which shows that the function Ψ, indeed, satisfies (2.11), and the corresponding function G solves the system (2.13).   ∎

Below we also establish a relationship between systems (2.13) and another type of the Riemann–Hilbert problems, where the function Ψ is allowed to have regular singularities.

(e) Relationship to isomonodromic deformations

The set of solutions of each system (2.13) has a subset corresponding to isomonodromic deformations of ordinary differential equations with meromorphic matrix coefficients,

$$\frac{\mathrm{d}\Psi}{\mathrm{d}\gamma} = A(\gamma)\Psi, \quad A(\gamma) = \sum_{j=1}^{L}\frac{A_j}{\gamma - z_j}, \tag{2.35}$$

where $A_j \in gl(r)$ are certain matrices which are independent of γ and such that $\sum_{j=1}^{L} A_j = 0$. Consider $GL(r)$-valued solution $\Psi(\gamma)$ of (2.35) satisfying the initial condition at some point $\gamma_0 \in \mathbb{C}P^1$,

$$\Psi(\gamma_0) = I. \tag{2.36}$$

The solution $\Psi(\gamma)$ has regular singularities at the points z_j; this function is generically non-single valued in the γ-plane: it gains the right multipliers M_j under analytical continuation along contours starting at γ_0 and encircling poles z_j. The matrices M_j are called the monodromy matrices of equation (2.35);

they generate the monodromy group of equation (2.35). If all the monodromy matrices are independent of the positions of singularities $\{z_j\}$, then (in the generic case, when none of the eigenvalues of each matrix A_j differ by an integer number) the function Ψ satisfies the following equations with respect to the positions of singularities z_j

$$\frac{\mathrm{d}\Psi}{\mathrm{d}z_j} = \left(\frac{A_j}{\gamma_0 - z_j} - \frac{A_j}{\gamma - z_j}\right)\Psi. \tag{2.37}$$

Compatibility of equations (2.35) and (2.37) is equivalent to the Schlesinger system for the functions $A_j(\{z_k\})$,

$$\frac{\partial A_j}{\partial z_k} = \frac{[A_j, A_k]}{z_j - z_k} - \frac{[A_j, A_k]}{\gamma_0 - z_k}, \quad j \neq k;$$

$$\frac{\partial A_j}{\partial z_j} = -\sum_{k \neq j}\left(\frac{[A_k, A_j]}{z_k - z_j} - \frac{[A_k, A_j]}{z_k - \gamma_0}\right). \tag{2.38}$$

The tau-function τ_{JM} of the Schlesinger system, introduced by Jimbo *et al.* (1981), is defined by the following system:

$$\frac{\partial}{\partial z_j}\ln\tau_{JM} = \frac{1}{2}\mathrm{res}\bigg|_{\gamma=z_j}\mathrm{tr}\left(\Psi_\gamma\Psi^{-1}\right)^2; \quad \frac{\partial\tau_{JM}}{\partial\bar{z}_j} = 0. \tag{2.39}$$

Each solution of the Schlesinger system induces a solution of hierarchy (2.13) according to the following theorem.

Theorem 2.3. *Consider a solution $\{A_j(\{z_k\})\}$ of the Schlesinger system (2.38), together with corresponding tau-function τ_{JM} and solution $\Psi(\gamma, \{z_j\})$ of the linear system (2.35) and (2.37) normalized by (2.36). Let $\mathcal{L}$ be a genus 0 covering of the λ-plane with simple ramification points $P_1, \ldots, P_M$; choose the map $\nu(P)$ to satisfy (2.1). Consider an arbitrary set of points P_0 and $Q_1, \ldots, Q_L$ on $\mathcal{L}$ such that their projections λ_0 and $\mu_1, \ldots, \mu_L$ on $\mathbb{C}P^1$, respectively, are independent of the branch points $\lambda_1, \ldots, \lambda_M$. Let us assume that the arguments of the solution and the tau-function of the Schlesinger system are given by the formulae*

$$z_j = \nu(Q_j), \quad \gamma_0 = \nu(P_0). \tag{2.40}$$

Then the function

$$\Psi(P, \{\lambda_m\}) = \Psi\left(\gamma, \gamma_0, \{z_j\}\right)\bigg|_{\gamma=\nu(P),\gamma_0=\nu(P_0),z_j=\nu(Q_j)}, \tag{2.41}$$

satisfies the linear system (2.11) of the hierarchy (2.13) with the functions J_m defined by

$$J_m(\{\lambda_n\}) = \frac{\alpha_m}{\gamma_m - \gamma_0}A(\gamma_m). \tag{2.42}$$

Therefore, according to (2.12), we have $J_m = G_{\lambda_m}G^{-1}$ for some function $G(\{\lambda_m\})$, which satisfies the system (2.13).

The corresponding tau-function τ is related to the Jimbo–Miwa tau-function as follows:

$$\tau(\{\lambda_m\}) = \prod_{j=1}^{L} \left(\frac{\partial \nu}{\partial \lambda}(Q_j)\right)^{\mathrm{tr}A_j^2/2} \tau_{JM}(\{z_j\})\bigg|_{z_j = \nu(Q_j)}, \tag{2.43}$$

where the derivative $\partial \nu / \partial \lambda$ is given by equation (2.2).

Proof. Taking the derivative of the function $\Psi(P)$ with respect to λ_m using the chain rule, we get

$$\frac{\partial \Psi(P)}{\partial \lambda_m}\bigg|_{\pi(P)} = \frac{\partial \Psi}{\partial \nu(P)} \frac{\partial \nu(P)}{\partial \lambda_m} + \sum_{j=1}^{L} \frac{\partial \Psi}{\partial z_j} \frac{\partial z_j}{\partial \lambda_m}$$

$$= \sum_{j=1}^{L} \frac{A_j}{\nu(P) - z_j} \frac{\alpha_m}{\gamma_m - \nu(P)} \Psi - \sum_{j=1}^{L} \frac{A_j}{\nu(P) - z_j} \frac{\alpha_m}{\gamma_m - z_j} \Psi$$

$$= -\frac{\alpha_m}{\nu(P) - \gamma_m} \sum_{j=1}^{L} \frac{A_j}{z_j - \gamma_m} \Psi = \frac{\gamma_0 - \gamma_m}{\nu(P) - \gamma_m} J_m \Psi,$$

where functions J_m are defined by (2.42).

Let us show how to prove the relation (2.43) between tau-functions. Taking into account the definition of Jimbo–Miwa tau-functions (2.39), we have

$$\frac{1}{2} \mathrm{tr}\left(\Psi_\gamma \Psi^{-1}\right)^2 = \sum_{j=1}^{L} \frac{\mathrm{tr}A_j^2}{2(\gamma - z_j)^2} + \sum_{j=1}^{L} \frac{\partial_{z_j} \ln \tau_{JM}}{\gamma - z_j}.$$

Now, using definition (2.16) of the tau-function τ, we get

$$\frac{\partial \ln \tau}{\partial \lambda_m} = \frac{\alpha_m}{2} \mathrm{tr}\left(\Psi_\nu \Psi^{-1}\right)^2\bigg|_{P=P_m}$$
$$= \sum_{j=1}^{L} \frac{\mathrm{tr}A_j^2}{2} \frac{\alpha_m}{(\gamma_m - z_j)^2} + \sum_{j=1}^{L} \frac{\alpha_m}{\gamma_m - z_j} \partial_{z_j} \ln \tau_{JM}. \tag{2.44}$$

By (2.2), we see that

$$\frac{\partial}{\partial \lambda_m} \ln\left\{\frac{\partial \nu}{\partial \lambda}(Q_j)\right\} = \frac{\alpha_m}{(\gamma_m - z_j)^2}, \quad \frac{\partial z_j}{\partial \lambda_m} = \frac{\alpha_m}{\gamma_m - z_j};$$

therefore, applying the chain rule in (2.44) (and taking into account that $\mathrm{tr}A_j^2$ are integrals of the Schlesinger system), we come to (2.43). ∎

Remark 2.1. The relationship between the systems (2.13) and isomonodromic deformations is a generalization of the link between the Ernst equation and the Schlesinger system established by Korotkin & Nicolai (1996).

(f) Rank 1 systems

When the function G from (2.13) is a scalar one, the system (2.13) can be rewritten as a system of *linear* scalar second-order differential equations in terms of the function $f(\{\lambda_m\}) = \ln G$,

$$(\gamma_m - \gamma_n)^2 \frac{\partial^2 f}{\partial \lambda_m \partial \lambda_n} - \alpha_n \frac{\gamma_m - \gamma_0}{\gamma_n - \gamma_0} \frac{\partial f}{\partial \lambda_m} - \alpha_m \frac{\gamma_n - \gamma_0}{\gamma_m - \gamma_0} \frac{\partial f}{\partial \lambda_n} = 0, \quad m \neq n. \quad (2.45)$$

In derivation of the system (2.45) from (2.13), we used equations (2.5).

In particular, any solution of the matrix system (2.13) gives a solution of the scalar system (2.45) if we put $f = \ln \det G$.

The linear system (2.11) turns in rank 1 into the scalar system

$$\left.\frac{\partial \psi(P)}{\partial \lambda_m}\right|_{\pi(P)} = \frac{\gamma_0 - \gamma_m}{\nu(P) - \gamma_m} \frac{\partial f}{\partial \lambda_m}, \quad (2.46)$$

where $\psi(P, \{\lambda_m\}) = \ln \Psi$. As well as in the matrix case, the function ψ is generically non-single valued on $\mathcal{L}$.

The definition of tau-function (2.16) now looks as follows:

$$\frac{\partial \ln \tau}{\partial \lambda_m} = \frac{(\gamma_0 - \gamma_m)^2}{2\alpha_m} \left\{\frac{\partial f}{\partial \lambda_m}\right\}^2. \quad (2.47)$$

Alternatively, it can be rewritten in terms of the function $\psi(P)$, using (2.19),

$$\frac{\partial \ln \tau}{\partial \lambda_m} = \frac{1}{2} \mathrm{res}\bigg|_{P_m} \left\{\frac{(\mathrm{d}\psi)^2}{\mathrm{d}\lambda}\right\}, \quad (2.48)$$

where $\mathrm{d}\psi(P) = \psi_\nu(P)\mathrm{d}\nu(P)$.

Let us discuss the solutions of system (2.46) and equation (2.45).

Theorem 2.4. *Let l be an arbitrary smooth closed contour on $\mathcal{L}$ such that its projection on the λ-plane $\pi(l)$ is independent of $\{\lambda_m\}$ and $P_m \notin l$ for any m. Consider on l an arbitrary Hölder-continuous function $h(P)$ independent of $\{\lambda_m\}$. Then the function*

$$f = \oint_l \frac{h(Q)\mathrm{d}\nu(Q)}{\nu(Q) - \gamma_0}, \quad (2.49)$$

satisfies system (2.45).

Proof. In the scalar case, we know explicitly the solution of an arbitrary Riemann–Hilbert problem from theorem 2.2. If we define the function

$$h(P) = \frac{1}{2\pi\mathrm{i}} \ln H(P).$$

(we assume that it is single valued on l, i.e. the index of $H(P)$ on l equals 0), the Riemann–Hilbert problem (2.32) becomes additive

$$\psi_+(P) = \psi_-(P) + 2\pi\mathrm{i}h(P), \quad (2.50)$$

for $P \in l$ (we recall that $\psi = \ln \Psi$); here $\psi_\pm$ stands for the boundary values of the function ψ on l. Normalization condition (2.33) turns into

$$\psi(\infty^{(1)}) = 0. \quad (2.51)$$

The solution of (2.50) and (2.51) for a Hölder-continuous function $h(P)$ is given by the Cauchy integral

$$\psi(P) = \oint_l \frac{h(Q)\mathrm{d}\nu(Q)}{\nu(Q) - \nu(P)}, \qquad (2.52)$$

which implies (2.49) at $P = P_0$, when $\nu(P) = \gamma_0$. ∎

Formula (2.49) can also be verified as follows. First, one can check by the direct substitution, using equations (2.2), (2.5) and (2.6), that the function

$$f = \frac{1}{\nu(Q) - \gamma_0} \frac{\partial \nu(Q)}{\partial \lambda}, \qquad (2.53)$$

satisfies system (2.45) for any $\lambda \equiv \pi(Q)$ independent of $\{\lambda_m\}$. Using linearity of equation (2.45), we can consider the superposition of these solutions at different $Q \in l$ with an arbitrary $\{\lambda_m\}$-independent measure $h(Q)$, which leads to (2.49).

In fact, we can consider any (say, compact) subset $D \subset \mathcal{L}$ whose projection on λ-plane is $\{\lambda_m\}$ independent, and such that $P_m \notin D$. We can define an arbitrary $\{\lambda_m\}$-independent measure $\mathrm{d}\mu(Q)$ on D; then the superposition principle implies that the function

$$f = \int_D \frac{h(Q)}{\nu(Q) - \gamma_0} \frac{\partial \nu(Q)}{\partial \lambda} \mathrm{d}\mu(Q), \qquad (2.54)$$

is a solution of (2.45).

(iii) *The Euler–Darboux equation*

As we noticed earlier, for the twofold covering with two branch points, $\lambda_1 = \xi$ and $\lambda_2 = \bar{\xi}$, the system (2.13) is equivalent to Ernst equation (2.22). In the scalar case, we get the following equation in terms of $f = \ln G$:

$$f_{\xi\bar{\xi}} - \frac{f_\xi - f_{\bar{\xi}}}{2(\xi - \bar{\xi})} = 0.$$

If we introduce the real coordinates (z, ρ) such that $\xi = z + \mathrm{i}\rho$, $\bar{\xi} = z - \mathrm{i}\rho$, this equation takes the form of Euler–Darboux equation,

$$f_{zz} + \frac{1}{\rho} f_\rho + f_{\rho\rho} = 0. \qquad (2.55)$$

From (2.8), taking into account that $P_0 = \infty^{(2)}$ and $\gamma_0 = (\lambda_1 + \lambda_2)/2$, we obtain

$$\frac{1}{\nu(P) - \gamma_0} \frac{\partial \nu(P)}{\partial \lambda} = \frac{1}{[(\lambda - \xi)(\lambda - \bar{\xi})]^{1/2}}.$$

Then representation (2.49) gives the solution (Courant & Hilbert 1962) of the Euler–Darboux equation,

$$f = \oint_l \frac{h(\lambda)\mathrm{d}\lambda}{[(\lambda - \xi)(\lambda - \bar{\xi})]^{1/2}}.$$

3. Generalization to higher genus

Here, we discuss possible ways to define systems, analogous to (2.13), starting from Hurwitz spaces $H_{g,N}$, which are spaces of meromorphic functions of degree N over Riemann surfaces of genus g, in genus $g \geq 2$ (we skip the case $g=1$ when the classification of stable bundles is rather special (Atiyah 1957), see also Remark 3.1 below).

As before, denote the critical points of a meromorphic function $\pi(P)$ on a Riemann surface of genus g by $P_1, \ldots, P_M$ and their images in $\mathbb{C}P^1$ by $\lambda_m = \pi(P_m)$. This meromorphic function realizes the Riemann surface as a ramified N-fold covering $\mathcal{L}$ of genus g of the λ-sphere with ramification points $P_1, \ldots, P_M$. Assuming that all the branch points λ_m are different and simple, we get, according to the Riemann–Hurwitz formula,

$$M = 2g + 2N - 2.$$

Let us introduce on $\mathcal{L}$ a canonical basis of cycles (a_α, b_α) $(\alpha = 1, \ldots, g)$ and corresponding basis of holomorphic 1-forms $w_\alpha(P)$ $(\alpha = 1, \ldots, g)$, normalized by $\oint_{a_\alpha} w_\beta = \delta_{\alpha\beta}$.

(a) Rank 1 systems in arbitrary genus

The scalar systems (2.45) admit natural generalization to the Hurwitz spaces $H_{g,N}$. Denote the prime form on $\mathcal{L}$ by $E(P, Q)$ (where $P, Q \in \mathcal{L}$); introduce the canonical meromorphic bidifferential $B(P, Q) = d_P d_Q \ln E(P, Q)$. By $B_m(P)$, we denote the meromorphic differential of second kind with vanishing a-periods and single pole at P_m of the second order with the following local behaviour:

$$B_m(P) = \left(\frac{1}{x_m^2} + O(1) \right) dx_m,$$

as $P \to P_m$, where $x_m = \sqrt{\lambda - \lambda_m}$ is a local parameter near P_m. Differentials B_m are related to $B(P, Q)$ as follows:

$$B_m(P) = \frac{B(P, Q)}{dx_m(Q)} \bigg|_{Q = P_m}. \tag{3.1}$$

Corresponding Abelian integrals we denote by Ω_m,

$$\Omega_m(P) = \int_{Q_0}^{P} B_m \equiv \frac{d}{dx_m(Q)} \left\{ \ln \frac{E(P, Q)}{E(Q_0, Q)} \right\} \bigg|_{Q = P_m}, \tag{3.2}$$

where $Q_0 \in \mathcal{L}$ is a base point such that its projection on λ-plane does not depend on $\{\lambda_n\}$.

Let us prove the following variational formula.

Theorem 3.1. *Assume that the local parameters x_P, x_Q do not depend on (some) λ_m. Introduce the symmetric function*

$$b(P, Q) = \frac{B(P, Q)}{dx_P dx_Q} \equiv \frac{\partial^2}{\partial x_P \partial x_Q} \ln E(P, Q). \tag{3.3}$$

Then dependence of $b(P, Q)$ on the branch point λ_m is given by the following equation:

$$\frac{\partial b(P, Q)}{\partial \lambda_m} = \frac{1}{2} b(P, P_m) b(Q, P_m). \tag{3.4}$$

Proof. Formula (3.4) is closely related to the Rauch variational formulae (Rauch 1959, Fay 1992) which describe dependence of holomorphic differentials on the branch points. The proof is also very similar. Namely, consider the λ_m-derivative of the canonical meromorphic bidifferential, $\partial B(P, Q)/\partial \lambda_m$. This is a symmetric 1-form on $\mathcal{L} \times \mathcal{L}$. Consider the Taylor series of $B(P, Q)$ with respect to its first argument P in a neighbourhood of ramification point P_m,

$$B(P, Q) = \{a_0 + a_1 x_m + \cdots\} \mathrm{d} x_m = \{a_0 + a_1 x_m + \cdots\} \frac{\mathrm{d}\lambda}{2\sqrt{\lambda - \lambda_m}}, \tag{3.5}$$

where $a_0, a_1, \ldots$ are some 1-forms with respect to Q. Differentiation of (3.5) with respect to λ_m gives

$$\left. \frac{\partial B(P, Q)}{\partial \lambda_m} \right|_{\pi(P)} = \{a_0 + o(1)\} \frac{\mathrm{d}\lambda}{4(\lambda - \lambda_m)^{3/2}} = \{a_0 + o(1)\} \frac{\mathrm{d} x_m}{2 x_m^2}, \tag{3.6}$$

as $P \to P_m$. Therefore, $\partial B(P, Q)/\partial \lambda_m$ is a meromorphic 1-form with respect to P, with the only pole at P_m of the second order with leading coefficient

$$\frac{a_0(Q)}{2} \equiv \left. \frac{B(Q, P)}{2 \mathrm{d} x_m(P)} \right|_{P = P_m}.$$

(which is itself a 1-form with respect to Q) and vanishing a-periods. Taking into account the symmetry of $B(P, Q)$, we get

$$\left. \frac{\partial B(P, Q)}{\partial \lambda_m} \right|_{\pi(P)} = \frac{1}{2} \left\{ \left. \frac{B(P, R)}{\mathrm{d} x_m(R)} \right|_{R = P_m} \right\} \left\{ \left. \frac{B(Q, R)}{\mathrm{d} x_m(R)} \right|_{R = P_m} \right\}, \tag{3.7}$$

which is equivalent to (3.4). ∎

Since differentials B_m are related to $B(P, Q)$ via (3.1), the theorem immediately implies the following useful.

Corollary 3.1. *The Abelian differential $B_n(P)$ and the Abelian integral $\Omega_n(P)$ depend on λ_m (for any $m \neq n$) as follows:*

$$\left. \frac{\partial}{\partial \lambda_m} B_n(P) \right|_{\pi(P)} = \frac{1}{2} b(P_m, P_n) B_m(P),$$

$$\left. \frac{\partial}{\partial \lambda_m} \Omega_n(P) \right|_{\pi(P)} = \frac{1}{2} b(P_m, P_n) \Omega_m(P). \tag{3.8}$$

Now we are in position to formulate the analogues of the Euler–Darboux equation in arbitrary genus.

Theorem 3.2. *Consider the following linear system of scalar equations for function $\psi(P, \{\lambda_m\})$:*

$$\left. \frac{\mathrm{d}\psi(P)}{\mathrm{d}\lambda_m} \right|_{\pi(P)} = R_m \Omega_m(P), \tag{3.9}$$

where R_m are some functions of $\{\lambda_m\}$. Denote by P_0 a point of $\mathcal{L}$ such that $\lambda_0 \equiv \pi(P_0)$ does not depend on $\{\lambda_m\}$. Then the compatibility conditions of the system (3.9) are given by the following equations:

$$\frac{\partial^2 f}{\partial \lambda_m \partial \lambda_n} - \frac{b(P_m, P_n)}{2} \left\{ \frac{v_n}{v_m} \frac{\partial f}{\partial \lambda_m} + \frac{v_m}{v_n} \frac{\partial f}{\partial \lambda_n} \right\} = 0, \qquad (3.10)$$

where $f(\{\lambda_m\}) = \psi(P_0)$ and

$$v_m \equiv \Omega_m(P_0) = \int_{Q_0}^{P_0} B_m;$$

function $b(P, Q)$ is defined by (3.3).

Proof. Substituting in (3.9) $P = P_0$, we see that

$$R_m = \frac{f \lambda_m}{\Omega_m(P_0)}.$$

Then the compatibility conditions of the system (3.9) are equivalent to the system of equations

$$\frac{\partial}{\partial \lambda_m} \left\{ \frac{\partial f}{\partial \lambda_n} \frac{\Omega_n(P)}{\Omega_n(P_0)} \right\} - \frac{\partial}{\partial \lambda_n} \left\{ \frac{\partial f}{\partial \lambda_m} \frac{\Omega_m(P)}{\Omega_m(P_0)} \right\} = 0, \quad m \neq n \qquad (3.11)$$

Using (3.8), we rewrite this equation as follows:

$$f_{\lambda_m \lambda_n} \left\{ \frac{\Omega_n(P)}{\Omega_n(P_0)} - \frac{\Omega_m(P)}{\Omega_m(P_0)} \right\}$$

$$+ \frac{b(P_m, P_n)}{2} \left\{ \Omega_m(P)\Omega_n(P_0) - \Omega_n(P)\Omega_m(P_0) \right\} \left\{ \frac{f_{\lambda_n}}{\Omega_n^2(P_0)} + \frac{f_{\lambda_m}}{\Omega_m^2(P_0)} \right\} = 0.$$

$$(3.12)$$

The l.h.s. of this equation is an Abelian integral on $\mathcal{L}$ with respect to argument P with vanishing a-periods (since all a-periods of differentials $B_m(P)$ vanish) and the poles (of the first order) only at P_m and P_n. Since $\Omega_m(P) = -1/x_m + O(1)$ as $P \to P_m$, we immediately see that equations (3.10) are equivalent to absence of poles of (3.12) at P_m and P_n. Therefore, the l.h.s. of (3.12) must be a constant with respect to P; choosing $P = P_0$ we see that this constant vanishes. We conclude that equations (3.10) indeed provide compatibility of the linear system (3.9). ∎

In analogy to the case of genus zero, now we shall define the tau-function of the system (3.10) and construct its solutions via solutions of the scalar Riemann–Hilbert problem on $\mathcal{L}$.

We define the tau-function of the system (3.10) by the following system of equations:

$$\frac{\partial}{\partial \lambda_m} \ln \tau = \frac{f_{\lambda_m}^2}{v_m^2}, \qquad \frac{\partial \ln \tau}{\partial \bar{\lambda}_m} = 0. \qquad (3.13)$$

From variational formulae (3.8) and equations (3.9), it follows that

$$\frac{\partial}{\partial\lambda_n}\left\{\frac{f_{\lambda_m}^2}{v_m^2}\right\} = b(P_m, P_n)\frac{f_{\lambda_m}f_{\lambda_n}}{v_m v_n}. \tag{3.14}$$

Symmetry of this expression with respect to m and n proves compatibility of equations (3.13).

As well as in the case of genus zero, we can prove the following

Lemma 3.1. *Definition (3.13) of the tau-function can be alternatively rewritten in the following form:*

$$\frac{\partial\ln\tau}{\partial\lambda_m} = \frac{1}{2}\text{res}\bigg|_{P_m}\left\{\frac{(\mathrm{d}_P\psi)^2}{\mathrm{d}\lambda}\right\}. \tag{3.15}$$

Proof. Choosing the standard local parameter $x_m = \sqrt{\lambda - \lambda_m}$ near P_m, we have $\mathrm{d}x_m = \mathrm{d}\lambda/2x_m$; therefore,

$$\text{res}\bigg|_{P_m}\left\{\frac{(\mathrm{d}_P\psi)^2}{\mathrm{d}\lambda}\right\} = \frac{1}{2}\left(\frac{\partial\psi}{\partial x_m}\right)^2(P_m).$$

On the other hand, as P belongs to a neighbourhood of P_m, the linear system (3.9) can be rewritten as follows:

$$\frac{\partial\psi}{\partial\lambda_m}\bigg|_{\nu(P)} - \frac{1}{2x_m}\frac{\partial\psi}{\partial x_m} = \frac{\partial f}{\partial\lambda_m}\frac{\Omega_m(P)}{\Omega_m(P_0)},$$

where we separated the dependence of ψ on λ_m which comes from λ_m-dependence of x_m. Taking the residue at P_m, we get

$$\frac{\partial\psi}{\partial x_m}\bigg|_{\nu(P)}(P_m) = 2\frac{f_{\lambda_m}}{\Omega_m(P_0)},$$

which implies coincidence of (3.13) and (3.15).   ∎

Theorem 3.3 provides solutions of the system (3.9).

Theorem 3.3. *Let l be an arbitrary smooth closed contour on $\mathcal{L}$ such that its projection on the λ-plane $\pi(l)$ is independent of $\{\lambda_m\}$ and $P_m \notin l$ for any m. Consider on l an arbitrary Hölder-continuous function $h(Q)$ independent of $\{\lambda_m\}$.*

Then the function

$$f = \oint_l h(Q)\mathrm{d}_Q\ln\frac{E(P_0, Q)}{E(Q_0, Q)}, \tag{3.16}$$

satisfies the system (3.10). Corresponding solution of the linear system (3.9) is given by

$$\psi(P) = \oint_l h(Q)\mathrm{d}_Q\ln\frac{E(P, Q)}{E(Q_0, Q)}. \tag{3.17}$$

Proof. Before proving the statement of the theorem, we note that if the function $h(Q)$ is Hölder continuous, then ψ (3.17) is a solution of the Riemann–Hilbert problem on the contour l with jump $2\pi i h(Q)$.

Let us now verify that the functions f and ψ defined by (3.16) and (3.17) satisfy the linear system (3.9),

$$\left.\frac{\partial\psi}{\partial\lambda_m}\right|_{\pi(P)} = \frac{\Omega_m(P)}{\Omega_m(P_0)}\frac{\partial f}{\partial\lambda_m}. \tag{3.18}$$

Taking into account representation (3.2) of $\Omega_m(P)$ in terms of prime forms, in analogy to Rauch formulae given by Th. 3.1, we see that the Cauchy kernel $d_Q\ln(E(P,Q)/E(Q_0,Q))$ depends on λ_m (assuming that all points P, Q, Q_0 are λ_m independent) as follows:

$$\frac{\partial}{\partial\lambda_m}\left\{d_Q\ln\frac{E(P,Q)}{E(Q_0,Q)}\right\} = \frac{1}{2}B_m(Q)\Omega_m(P). \tag{3.19}$$

Therefore,

$$\frac{\psi_{\lambda_m}(P)}{\Omega_m(P)} = \frac{1}{2}\oint_l h(Q)B_m(Q).$$

Independence of this expression of P proves (3.18). Therefore, the function f satisfies the compatibility conditions of (3.18), i.e. the equations (3.10). ∎

We note that, using superposition principle for the systems (3.10), we can substitute integration over contour in (3.16) by integration with respect to an arbitrary $\{\lambda_m\}$-independent measure with compact support; the only condition we need to impose on this measure is commutativity of integration in (3.16) with differentiation with respect to $\{\lambda_m\}$.

(b) Systems of higher rank

Here we consider coverings of genus $g \geq 2$. To define rank r analogues of equations (3.10), it is natural to start from generalization of the zero curvature representation (2.11) to higher genus,

$$\frac{\partial\Psi}{\partial\lambda_m} = U_m\Psi, \tag{3.20}$$

where the $r \times r$ 'Lax matrix' $U_m(P, \{\lambda_n\})$, $P \in \mathcal{L}$ has only one singularity on $\mathcal{L}$, which is a simple pole at the ramification point P_m. We shall assume that $\operatorname{tr} U_m = 0$ (this condition can be always satisfied by normalization of function Ψ, $\Psi \to [\det\Psi]^{-1/r}\Psi$). As well as in rank 1, the Lax matrix $U_m(P)$ cannot be single valued on $\mathcal{L}$. In rank 1 case, U_m has additive twists along basic cycles on $\mathcal{L}$, but $d_P U_m(P)$ is a single-valued meromorphic differential on $\mathcal{L}$ with pole of second order at P_m.

In higher rank, this is not enough—it is necessary to introduce more degrees of freedom, and allow $d_P U_m$ to suffer similarity transformation under tracing along each topologically non-trivial closed contour on $\mathcal{L}$ (this transformation may be $\{\lambda_m\}$ dependent itself).

More precisely, consider a stable vector bundle χ of rank r and degree d over $\mathcal{L}$. Require $d_P U_m$ to be a meromorphic section of the bundle $\operatorname{ad}_\chi \otimes K$ with quadratic pole at the ramification point P_m (K is the canonical line bundle), i.e.

$$d_P U_m \in H^0(\mathcal{L}, \operatorname{ad}_\chi \otimes K(2P_m));$$

according to terminology proposed by Hitchin (1987), $d_P U_m$ is called the meromorphic Higgs field.

According to the Narasimhan–Seshadri theorem (Narasimhan & Seshadri 1965), a stable vector bundle χ is characterized by the set of unitary constant (independent of a point $P \in \mathcal{L}$) matrices $\chi_{a_1}, ..., \chi_{a_g}, \chi_{b_1}, ..., \chi_{b_g}$, satisfying one relation

$$\prod_{\alpha=1}^{g} \chi_{a_j} \chi_{b_j} \chi_{a_j}^{-1} \chi_{b_j}^{-1} = \exp\left\{2\pi i \frac{d}{r}\right\}.$$

Stability of the bundle χ implies $h^0(ad_\chi) = 0$, i.e. the bundle ad_χ does not have any holomorphic section. On the other hand, $h^0(ad_\chi \otimes K) = (g-1)(r^2-1)$; therefore, fixing the singular part of $d_P U_m$ at the ramification point P_m,

$$d_P U_m(P) = \left(-\frac{R_m}{x_m^2} + O(1)\right) dx_m \text{ as } P \to P_m, \tag{3.21}$$

we define $d_P U_m$ up to an arbitrary linear combination of $(g-1)(r^2-1)$ holomorphic sections of $ad_\chi \otimes K$. In rank 1 case, we fixed $d_P U_m$ uniquely imposing the normalization condition of vanishing of all a-periods of $d_P U_m$; unfortunately, such simple normalization in higher rank is not known to the authors.

The Lax matrix U_m is uniquely defined by $d_P U_m$ inside of the fundamental polygon $\hat{\mathcal{L}}$ of $\mathcal{L}$ if we fix a normalization point $P_0 \in \hat{\mathcal{L}}$ and assume that $U_m(P_0) = 0$. The only singularity of the matrix $g_{mm} = \{(1/2)\oint_l h(Q) B_m(Q)\}^2$ is the simple pole at P_m of the form

$$U_m(P) = \frac{R_m}{x_m} + O(1) \text{ as } P \to P_m;$$

generically U_m has both multiplicative and additive non-single valuedness along any cycle $\gamma \in \pi_1(\mathcal{L})$,

$$U_m(P^\gamma) = \chi_\gamma U_m(P) \chi_\gamma^{-1} + C_m^\gamma; \tag{3.22}$$

since $d_P U_m(P^\gamma) = \chi_\gamma d_P U_m(P) \chi_\gamma^{-1}$, the 'additive twists' C_m^γ do not depend on P.

The compatibility conditions of the linear system (3.20) are given by

$$F_{mn}(P) = 0, \tag{3.23}$$

where $F_{mn}(P)$ is the curvature,

$$F_{mn}(P) = \frac{\partial U_m}{\partial \lambda_n} - \frac{\partial U_n}{\partial \lambda_m} + [U_m, U_n]. \tag{3.24}$$

To rewrite the compatibility conditions (3.23) in terms of variables depending only on $\{\lambda_m\}$ (and not on the point P of the covering $\mathcal{L}$), we consider the coefficients of the Taylor series of $U_m(P)$ at the ramification point P_n, $n \neq m$,

$$U_m(P) = S_{mn} + T_{mn} x_n + O(x_n^2).$$

Then the non-singularity of F_{mn} at P_m is equivalent to the condition

$$\frac{\partial R_m}{\partial \lambda_n} + \frac{1}{2} T_{nm} + [R_m, S_{nm}] = 0, \, m \neq n. \tag{3.25}$$

The non-singularity of F_{mn} is insufficient for its vanishing, since $F_{mn}(P)$ has both multiplicative and additive twists along topologically non-trivial loops. However, if we require that the additive twists $\Gamma_{nm}^n = \beta_{mn}(\oint h(P) B_m(P))/(\oint h(P) B_n(P))$

from (3.22) are related to matrices χ_γ via equations

$$\frac{\partial \chi_\gamma}{\partial \lambda_m} \chi_\gamma^{-1} = C_m^\gamma, \tag{3.26}$$

we observe that transformation (3.22) of the Lax matrices U_m is nothing but the gauge transformation of connection 1-form $\sum_{m=1}^M U_m \mathrm{d}\lambda_m$ by the matrix $\chi_\gamma(\{\lambda_m\})$. Then the curvature coefficient F_{mn} transforms as follows:

$$F_{mn}(P^\gamma) = \chi_\gamma F_{mn}(P) \chi_\gamma^{-1},$$

i.e. $F_{mn} \in H^0(ad_\chi)$; thus $F_{mn} = 0$.

We conclude that the compatibility conditions of the linear system (3.20) in arbitrary rank r and any genus $g \geq 2$, are given by the system of equations (3.25) and (3.26). The new feature in comparison with the genus zero case is that we get another degree of freedom—the stable bundle χ must itself depend on the branch points λ_m, according to equations (3.25) (since the generic vector bundle over $\mathcal{L}$ is stable, generically evolution (3.26) preserves stability of χ at least locally, in a neighbourhood of a given stable bundle).

Obviously, without any normalization of the twisted 1-form $\mathrm{d}_P U_m$, the coefficients S_{mn}, T_{mn}, and additive twists C_m^γ, are not uniquely determined by the set of residues R_m, and matrices χ_γ. Therefore, the number of equations (3.25) and (3.26) is substantially smaller than the number of variables

A possible way to define $\mathrm{d}_P U_m$, uniquely is to make use of one of meromorphic bidifferentials $W(P, Q)$, on $\mathcal{L} \times \mathcal{L}$ whose existence is provided by the following lemma

Lemma 3.2. *There exists meromorphic bidifferential $W(P, Q)$. on $\mathcal{L} \times \mathcal{L}$ satisfying the following conditions.*

(i) *On the diagonal $P = Q$, the bidifferential $W(P, Q)$ has second-order pole with biresidue equal to $r^2 \times r^2$ matrix Π (which is the permutation matrix in $\mathbb{C}^r \otimes \mathbb{C}^r$),*

$$W(P, Q) = \left\{ \frac{\Pi}{(x_P - x_Q)^2} + O(1) \right\} dx_P \, dx_Q. \tag{3.27}$$

(ii) *Symmetry condition:*

$$W(P, Q) = \Pi W(Q, P) \Pi. \tag{3.28}$$

(iii) *Automorphy conditions: for any $\gamma \in \pi_1(\mathcal{L})$, we have*

$$W(P^\gamma, Q) = \overset{1}{\chi}_\gamma W(P, Q) \overset{1}{\chi}_\gamma^{-1}, \tag{3.29}$$

$$W(P, Q^\gamma) = \overset{2}{\chi}_\gamma W(P, Q) \overset{2}{\chi}_\gamma^{-1}, \tag{3.30}$$

where for any linear operator A in $\mathbb{C}^r$, we denote by A^1 and A^2 the operators $A \otimes I$ and $I \otimes A$ in $\mathbb{C}^r \otimes \mathbb{C}^r$, respectively.

The relation (3.30) is obviously a corollary of (3.29) and the symmetry requirement (3.28). Equivalently, relations (3.29) and (3.30) mean that $W(P, Q)$ belongs

to $H^0(ad_\chi^1 \otimes K(2Q))$ with respect to its first argument, and to $H^0(ad_\chi^2 \otimes K(2P))$ with respect to its second argument.

Proof. Existence of bidifferential $W_0(P, Q)$ satisfying only (3.27) and automorphy properties (3.29) and (3.30) can be proved similarly to existence of Schiffer kernel corresponding to the bundle ad_χ (e.g. Fay (1992)). To construct bidifferential $W(P, Q)$ which satisfies in addition the symmetry condition (3.28), suppose that $F(P, Q) \equiv W_0(P, Q) - \Pi W_0(Q, P)\Pi$ does not vanish; this is a holomorphic section of $ad_\chi^1 \otimes K$ with respect to P and holomorphic section of $ad_\chi^2 \otimes K$ with respect to Q. Obviously,

$$\Pi F(P, Q)\Pi = -F(Q, P).$$

Now define $W(P, Q) = W_0(P, Q) - \frac{1}{2}F(P, Q)$. Simple calculation shows that it satisfies (3.28); other required properties are inherited from $W_0(P, Q)$. ∎

We proved existence of $W(P, Q)$; obviously this bidifferential is not unique: we can add to $W(P, Q)$ an arbitrary linear combination of bilinear products of holomorphic sections $f_k(P)$ of $ad_\chi \otimes K$, satisfying the symmetry condition (3.28); this is a linear combination of the form

$$\sum_{j,k=1}^{(g-1)(r^2-1)} \alpha_{jk} \left\{ \overset{1}{f_k}(P)\overset{2}{f_j}(Q) + \overset{1}{f_j}(P)\overset{2}{f_k}(Q) \right\},$$

with arbitrary $\alpha_{jk} \in \mathbb{C}$.

Let us fix the bidifferential $W(P, Q)$ in some way (e.g. according to Remark 3.2 below). Then the Higgs fields $d_P U_m$ can be defined as follows:

$$d_P U_m(P) = \frac{\operatorname{tr}\left\{ \overset{12}{W}(P, Q)\overset{2}{R_m} \right\}}{dx_m(Q)} \Bigg|_{Q=P_m}. \tag{3.31}$$

Now all variables S_{mn}, T_{mn}, C_m^γ become functionals of the residues R_m, branch points λ_m and matrices χ_γ; the number of variables (χ_γ, R_m) in the system (3.25) and (3.26) coincides with the number of equations.

The natural definition of the tau-function, in agreement with (2.19), looks as follows:

$$\frac{\partial}{\partial \lambda_m} \ln \tau = \frac{1}{2} \operatorname{res}\Bigg|_{P_m} \frac{\operatorname{tr}(d_P \Psi \Psi^{-1})^2}{d\lambda}. \tag{3.32}$$

Consistency of the definition (3.32) is provided by the following

Lemma 3.3. Let function Ψ solve the linear system (3.20) with Lax matrices U_m satisfying condition $U_m(P_0) = 0$ for some $P_0 \in \mathcal{L}$, and conditions (3.31). Then the 1-form

$$\sum_{m=1}^{M} \left\{ \operatorname{res}\Bigg|_{P_m} \frac{\operatorname{tr}(d_P \Psi \Psi^{-1})^2}{d\lambda} \right\} d\lambda_m, \tag{3.33}$$

is closed.

Proof. Consider the Taylor series of $\Psi(P)$ in a neighbourhood of P_m,

$$\Psi(P) = \Psi_0 + x_m \Psi_1 + O(x_m^2);$$

then

$$\Psi_{x_m}\Psi^{-1} = \Psi_1\Psi_0^{-1} + O(1),$$

and

$$\Psi_{\lambda_m}\Psi^{-1} = -\frac{1}{2x_m}\Psi_1\Psi_0^{-1} + O(1).$$

Therefore,

$$R_m = -\frac{1}{2}\Psi_1\Psi_0^{-1},$$

and

$$\frac{1}{2}\mathrm{res}\bigg|_{P_m}\frac{\mathrm{tr}(\mathrm{d}_P\Psi\Psi^{-1})^2}{\mathrm{d}\lambda} = \frac{1}{2}\mathrm{res}\bigg|_{P_m}\frac{\mathrm{tr}(-2R_m\mathrm{d}x_m)^2}{2x_m\mathrm{d}x_m} = \mathrm{tr}R_m^2.$$

Now we have to make sure that the derivative $\partial(\mathrm{tr}R_m^2)/\partial\lambda_n$ is symmetric under the interchanging of m and n,

$$\frac{\partial}{\partial\lambda_n}\mathrm{tr}R_m^2 = -2\mathrm{tr}R_m\left\{\frac{1}{2}T_{nm} + [R_m, S_{nm}]\right\} = -\mathrm{tr}R_m T_{nm}. \qquad (3.34)$$

As $P \to P_m$, we have

$$\mathrm{tr}\{\mathrm{d}_P U_m(P)\mathrm{d}_P U_n(P)\} = (-\mathrm{tr}R_m T_{nm} + \cdots)\left(\frac{\mathrm{d}x_m}{x_m}\right)^2;$$

symmetry of (3.34) is thus equivalent to relation

$$\mathrm{bires}\bigg|_{P_m}\mathrm{tr}\{\mathrm{d}_P U_m\mathrm{d}_P U_n\} = \mathrm{bires}\bigg|_{P_n}\mathrm{tr}\{\mathrm{d}_P U_m\mathrm{d}_P U_n\}.$$

Let us rewrite the l.h.s. of this relation in terms of bidifferential $W(P,Q) \equiv w(P,Q)\mathrm{d}x_P\mathrm{d}x_Q$ according to (3.31), taking into account the behaviour (3.27) of $W(P,Q)$ on the diagonal $P=Q$,

$$\mathrm{bires}\bigg|_{P_m}\mathrm{tr}\{\mathrm{d}_P U_m\mathrm{d}_P U_n\} = \mathrm{bires}\bigg|_{P_m}\overset{1}{\mathrm{tr}}\left\{\overset{3}{\mathrm{tr}}\left\{\overset{3}{R}_m\overset{13}{w}(P,P_m)\right\}\overset{2}{\mathrm{tr}}\left\{\overset{2}{R}_n\overset{12}{w}(P,P_n)\right\}\right\}(\mathrm{d}x_m(P))^2$$

$$= \overset{1}{\mathrm{tr}}\overset{2}{\mathrm{tr}}\left\{\overset{1}{R}_m\overset{2}{R}_n\overset{12}{w}(P_m,P_n)\right\}. \qquad (3.35)$$

Similarly,

$$\mathrm{bires}\bigg|_{P_m}\mathrm{tr}\{\mathrm{d}_P U_m\mathrm{d}_P U_n\} = \overset{1}{\mathrm{tr}}\ \overset{2}{\mathrm{tr}}\ \left\{\overset{2}{R}_m\overset{1}{R}_n\overset{12}{w}\ (P_n,P_m)\right\}$$

$$= \overset{1}{\mathrm{tr}}\ \overset{2}{\mathrm{tr}}\ \left\{\overset{2}{R}_m\overset{1}{R}_n\Pi\overset{12}{w}\ (P_m,P_n)\Pi\right\}$$

$$= \overset{1}{\mathrm{tr}}\ \overset{2}{\mathrm{tr}}\ \left\{\overset{1}{R}_m\overset{2}{R}_n\overset{12}{w}\ (P_m,P_n)\right\}, \qquad (3.36)$$

coinciding with (3.35); the last equality in (3.36) follows from the symmetry property (3.28) of $W(P,Q)$. ∎

Remark 3.1. More explicit treatment is, as usual, possible for elliptic coverings $g = 1$, when the bundle ad_χ can possess meromorphic sections with single simple pole. In this case, the additive twists C_m^γ are absent, and monodromy matrices χ_γ of the bundle χ can be chosen to be independent of $\{\lambda_m\}$. In this case, the above construction can be nicely rewritten in terms of the elliptic r-matrix (Shramchenko 2003).

Remark 3.2. In rank 1 case, we have fixed the bidifferential $W(P, Q)$ by the requirement that all of its a-periods vanish; then $W(P, Q)$ coincides with canonical meromorphic bidifferential. This kind of normalization is not possible in higher rank due to non-invariance of $W(P, Q)$ under tracing along homologically non-trivial loops on $\mathcal{L}$. However, in rank 1 case, there exists another way to fix $W(P, Q)$ uniquely: one can require that $\text{v.p.}_\mathcal{L} W(P, Q) w_\alpha(Q) = 0$ for any holomorphic differential w_α (in this case $W(P, Q)$ is called the Schiffer kernel). These conditions have natural higher-rank analogue (Fay 1992). Namely, we can require that $\text{v.p.}_\mathcal{L} \text{tr}^2 \{ W^{12}(P, Q) f_k^{\dagger^2}(Q) \} = 0$ for any $f_k \in H^0(ad_\chi \otimes K)$; due to unitarity of the matrices χ_α the integrand is a $(1, 1)$-form on $\mathcal{L}$. If the biresidue at $P = Q$ is chosen to be the unit matrix $I \otimes I$ instead of permutation matrix Π, our definition of $W(P, Q)$ would give the Schiffer kernel corresponding to the bundle ad_χ (Fay 1992). This normalization of $W(P, Q)$ leads to the following normalization of $d_P U_m$: $\text{v.p.}_\mathcal{L} \text{tr} \{ d_P U_m f_k^\dagger \} = 0$ for any $f_k \in H^0(ad_\chi \otimes K)$.

We note that, in contrast to canonical meromorphic bidifferential in rank 1, this normalization of $W(P, Q)$ makes it non-holomorphic function of 'moduli' λ_m; therefore, in principle, the complete system of equations should contain also equations with respect to $\{\overline{\lambda_m}\}$. We shall discuss these aspects in more details in further publication.

(i) *Relationship to isomonodromy deformations in higher genus*

The link between the genus zero systems (2.13) and isomonodromic deformations on Riemann sphere discussed in §2e can be extended to arbitrary genus (we outline this link here for $g \geq 2$). Let us briefly describe the isomonodromic deformations on a Riemann surface $\mathcal{L}$. Consider a stable flat vector bundle χ characterized by a set of unitary matrices χ_γ for any $\gamma \in \pi_1(\mathcal{L})$; consider also a divisor $Q = Q_1 + \dots + Q_L$ on $\mathcal{L}$.

Introduce a Higgs field $A(P)$ which is allowed to have simple poles at the points $Q_1, \dots, Q_L$, i.e. $A \in H^0(ad_\chi \otimes K(Q))$; suppose that $\text{tr} A(P) = 0$. The higher-genus analogue of the linear differential equation (2.35) looks as follows:

$$d_P \Psi = A(P) \Psi, \qquad (3.37)$$

where function Ψ has unit determinant and satisfies the initial condition $\Psi(P_0) = I$ at some point $P_0 \in \mathcal{L}$.

In analogue to genus 0 case, function Ψ has regular singularities at the points $Q_1, \dots, Q_L$ with some monodromy matrices M_k, i.e. under tracing around Q_k the function $\Psi(P)$ transforms as follows:

$$\Psi(P) \to \Psi(P) M_k.$$

Under tracing along basic cycles of $\mathcal{L}$, the function Ψ gains left multipliers given by the matrices χ^γ; in addition, it may gain the right multipliers M_{a_α}, M_{b_α}, which

are analogues of monodromy matrices M_k,

$$\Psi(P^{a_\alpha}) = \chi_{a_\alpha}\Psi(P)M_{a_\alpha}, \quad \Psi(P^{b_\alpha}) = \chi_{b_\alpha}\Psi(P)M_{b_\alpha}.$$

The monodromy matrices M_{a_α}, M_{b_α} generate a $SL(r)$ 'monodromy' representation of the fundamental group $\pi_1(\mathcal{L}\backslash\{Q_k\})$.

If we now assume that all the monodromy matrices $M_k, M_{a_\alpha}, M_{b_\alpha}$ are independent of the branch points λ_m and projections $\mu_k = \pi(Q_k)$ of the regular singularities, then function Ψ satisfies the deformation equations

$$\Psi_{\mu_k} = V_k\Psi, \tag{3.38}$$

$$\Psi_{\lambda_m} = U_m\Psi, \tag{3.39}$$

where matrix V_k has simple pole at the regular singularity Q_k; matrix U_m has simple pole at the ramification point P_m; these matrices transform as follows under the tracing along any $\gamma \in \pi_1(\mathcal{L})$:

$$V_k(P^\gamma) = \chi_\gamma V_k(P)\chi_\gamma^{-1} + (\chi_\gamma)_{\mu_k}\chi_\gamma^{-1},$$

$$U_m(P^\gamma) = \chi_\gamma U_m(P)\chi_\gamma^{-1} + (\chi_\gamma)_{\lambda_m}\chi_\gamma^{-1}.$$

Obviously, the part (3.39) of the deformation equations is nothing but the linear system (3.20) introduced above; therefore, the isomonodromic deformations in higher genus correspond, as well as in genus zero, to a subset of solutions of the integrable systems (3.25) and (3.26).

4. Systems of rank 1, Darboux–Egoroff metrics and systems of hydrodynamic type

(a) Darboux–Egoroff metrics

It turns out that each solution of the rank 1 system (3.10) defines a flat diagonal (pseudo-)metric in $\mathbb{C}^M$ (which gives rise to a flat diagonal Darboux–Egoroff metric in $\mathbb{R}^M$ if additional reality and positivity conditions are imposed). In the sequel, we shall (in agreement with previous works on the subject) use the term 'Darboux–Egoroff metric' for such pseudo-metrics in $\mathbb{C}^M$.

For diagonal (pseudo-)metric

$$ds^2 = \sum_{m=1}^{M} g_{mm}d\lambda_m^2, \tag{4.1}$$

the Christoffel symbols are given by

$$\Gamma_{mn}^k = 0, \Gamma_{nm}^n = \partial_{\lambda_m}\ln\sqrt{g_{nn}}, \quad m \neq n \neq k; \tag{4.2}$$

they are related to rotation coefficients β_{mn} as follows:

$$\beta_{mn} = \frac{\sqrt{g_{mm}}}{\sqrt{g_{nn}}}\Gamma_{mn}^m \equiv \frac{\partial_{\lambda_n}\sqrt{g_{mm}}}{\sqrt{g_{nn}}}, \quad m \neq n. \tag{4.3}$$

The metric (4.1) is flat iff the rotation coefficients satisfy the following equations:

$$\frac{\partial\beta_{mn}}{\partial\lambda_l} = \beta_{ml}\beta_{ln}, \tag{4.4}$$

for any distinct l, m, n, and each of β_{mn} is invariant with respect to simultaneous

shifts along all $\{\lambda_k\}$,

$$\sum_{k=1}^{M} \frac{\partial \beta_{mn}}{\partial \lambda_k} = 0. \tag{4.5}$$

If in addition the rotation coefficients are symmetric, $\beta_{mn} = \beta_{nm}$, which is equivalent to the relation $\partial_m g_{nn} = \partial_n g_{mm}$, then there exists potential U such that

$$g_{mm} = \frac{\partial U}{\partial \lambda_m},$$

and the metric (4.1) is called the Darboux–Egoroff metric.

We shall now prove that each solution of the system (3.10) corresponds to Darboux–Egoroff metric.

Theorem 4.1. *Let f be an arbitrary solution of the system (3.9), and $\tau(\{\lambda_m\})$ be the corresponding tau-function defined by (3.13) or (3.15). Then metric (4.1) with*

$$g_{mm} = \frac{\partial \ln \tau}{\partial \lambda_m}, \tag{4.6}$$

is a Darboux–Egoroff metric in $\mathbb{C}^M$.

Proof. Let us compute the rotation coefficients of the metric (4.1). From the definition of the tau-function (3.13), we have

$$\sqrt{\frac{\partial \ln \tau}{\partial \lambda_m}} = \frac{f_{\lambda_m}}{v_m};$$

using the expression (3.14) for the second derivative of the tau-function, we find

$$\beta_{mn} = \frac{1}{2} \frac{(\ln \tau)_{\lambda_m \lambda_n}}{\sqrt{(\ln \tau)_{\lambda_m}} \sqrt{(\ln \tau)_{\lambda_n}}} = \frac{1}{2} b(P_m, P_n). \tag{4.7}$$

These functions satisfy the equations (4.4) as a corollary of variational formulae (3.4).

It remains to prove that each $b(P_m, P_n)$ satisfies equations (4.5). We shall use invariance of the canonical meromorphic bidifferential $B(P, Q)$ with respect to biholomorphic maps. Let us consider the branched covering $\mathcal{L}^\epsilon$, which is obtained by small ϵ-shift of all the ramification points P_m in λ-plane, i.e. the projections of branch points P_m^ϵ of $\mathcal{L}^\epsilon$ on λ-plane are equal to $\lambda_m^\epsilon = \lambda_m + \epsilon$; B^ϵ is the canonical meromorphic bidifferential on $\mathcal{L}^\epsilon$. Denote the projections of points P and Q on the λ-plane by λ and μ, respectively. Define the point P^ϵ to be the point lying on the same sheet as P and having projection $\lambda + \epsilon$ on the λ-plane; in the same way, point Q^ϵ belongs to the same sheet as Q and has projection $\mu + \epsilon$ on the λ-plane. Since $\mathcal{L}^\epsilon$ can be holomorphically mapped to $\mathcal{L}$ by transformation $\lambda \rightarrow \lambda + \epsilon$ on all the sheets, we have

$$B^\epsilon(P^\epsilon, Q^\epsilon) = B(P, Q). \tag{4.8}$$

Assuming that P belongs to a neighbourhood of the branch point P_m, and Q belongs to a neighbourhood of the branch point P_n, we can write down the respective local parameters as $x_m(P) = \sqrt{\lambda - \lambda_m}$ and $x_n(Q) = \sqrt{\mu - \lambda_n}$. These parameters are obviously invariant with respect to simultaneous ϵ-shifts of all $\{\lambda_m\}$, λ and μ: $x_m^\epsilon(P^\epsilon) = x_m(P)$ and $x_n^\epsilon(Q^\epsilon) = x_n(Q)$. Therefore, equality (4.8)

induces the same relation between $b(P, Q) \equiv B(P, Q)/\mathrm{d}x(P)\mathrm{d}x(Q)$:

$$b^{\epsilon}(P^{\epsilon}, Q^{\epsilon}) = b(P, Q). \tag{4.9}$$

Assuming now that $P = P_m$ and $Q = P_n$ and differentiating (4.9) with respect to ϵ at $\epsilon = 0$, we come to (4.5). ∎

Of course, besides simultaneous translations, the canonical meromorphic bidifferential is invariant with respect to any other Möbius transformation of the λ-plane performed simultaneously on all the sheets of $\mathcal{L}$. We can use this invariance to obtain two relations, corresponding to other one-parametric families of Möbius transformations.

Proposition 4.1. *The rotation coefficients (4.7) of the Darboux–Egoroff metric (4.1) satisfy the following relations*:

$$\left\{ \sum_{k=1}^{M} \lambda_k \frac{\partial}{\partial \lambda_k} \right\} \beta_{mn} = -\beta_{mn}, \tag{4.10}$$

$$\left\{ \sum_{k=1}^{M} \lambda_k^2 \frac{\partial}{\partial \lambda_k} \right\} \beta_{mn} = -(\lambda_n + \lambda_m)\beta_{mn}. \tag{4.11}$$

Proof. Relation (4.10) corresponds to invariance of the canonical meromorphic bidifferential under simultaneous dilatation on every sheet of $\mathcal{L}$, i.e. to the transformations

$$\lambda_m \to (1 + \epsilon)\lambda_m, \quad \lambda \to (1 + \epsilon)\lambda, \quad \mu \to (1 + \epsilon)\mu.$$

The new feature in comparison with the proof of relation (4.5) is that the local parameters are now dependent on ϵ,

$$x_m^{\epsilon}(P^{\epsilon}) \equiv [(1 + \epsilon)\lambda - (1 + \epsilon)\lambda_m]^{1/2} = (1 + \epsilon)^{1/2}x_m(P),$$

and

$$x_n^{\epsilon}(Q^{\epsilon}) \equiv [(1 + \epsilon)\mu - (1 + \epsilon)\lambda_n]^{1/2} = (1 + \epsilon)^{1/2}x_n(Q);$$

therefore, invariance $B^{\epsilon}(P^{\epsilon}, Q^{\epsilon}) = B(P, Q)$ of the canonical meromorphic bidifferential translates on the level of $b(P, Q)$ as follows:

$$(1 + \epsilon)b^{\epsilon}(P^{\epsilon}, Q^{\epsilon}) = b(P, Q). \tag{4.12}$$

Differentiating this relation with respect to ϵ at $\epsilon = 0$ via the chain rule and choosing $P = P_m$ and $Q = P_n$, we get (4.10).

In a similar way, we can deduce (4.11) from invariance of the canonical meromorphic bidifferential with respect to the one-parametric family of transformations

$$\lambda \to \lambda^{\epsilon} = \frac{\lambda}{1 + \epsilon\lambda},$$

on each sheet of $\mathcal{L}$. ∎

Relation (4.10) for the rotation coefficients can be found in Dubrovin (1996); relation (4.11) seems to be new. We also note that all primary differentials used by Dubrovin (1996) to construct Frobenius manifolds from Hurwitz spaces can be obtained from solutions (3.16) of the systems (3.10) by appropriate specification of the contour l and the function $h(Q)$.

(b) *Systems of hydrodynamic type*

According to well-known results (e.g. review by Tsarev (1990)), to each Darboux–Egoroff metric, one can associate a class of diagonal systems of hydrodynamic type

$$\frac{\partial \lambda_m}{\partial x} = V_m \frac{\partial \lambda_m}{\partial t}, \tag{4.13}$$

where functions $V_m(\{\lambda_k\})$ (the 'characteristic speeds') are related to Christoffel symbols (4.2) of the metric (4.1) via system of differential equations,

$$\partial_m V_n = \Gamma^n_{nm}(V_m - V_n). \tag{4.14}$$

Compatibility of equations (4.14) is provided by equations (4.4) for rotation coefficients.

Let us choose some solution of the system (3.10) parametrized by an arbitrary function $h(P)$ on contour l (3.16). Then the metric coefficients (4.6) are given by

$$g_{mm} = \left\{ \frac{1}{2} \oint_l h(Q) B_m(Q) \right\}^2. \tag{4.15}$$

The Christoffel coefficients of this metric look as follows:

$$\Gamma^n_{nm} = \beta_{mn} \frac{\displaystyle\oint_l h(P) B_m(P)}{\displaystyle\oint_l h(P) B_n(P)}. \tag{4.16}$$

Solutions of equations (4.14) for these Christoffel coefficients are described by the following proposition[2]

Proposition 4.2. *Let $h_1(P)$ be an arbitrary Hölder-continuous function and independent of $\{\lambda_m\}$ function on contour l. Then the functions*

$$V_m = \frac{\displaystyle\oint_l h_1(P) B_m(P)}{\displaystyle\oint_l h(P) B_m(P)}, \tag{4.17}$$

satisfy system (4.14) with Christoffel coefficients given by (4.2).

The proof of this proposition is a simple calculation based on Rauch variational formulae for differentials B_m,

$$\frac{\partial}{\partial \lambda_n} B_m(P) = \beta_{mn} B_n(P). \tag{4.18}$$

To construct solutions of the system (4.13) with characteristic speeds (4.17), one needs to use the following theorem by Tsarev (1990):

Theorem 4.2. *Let functions $V_m(\lambda_1, \ldots, \lambda_M)$ satisfy equations (4.14). Then the system of equations*

$$\Phi_m(\{\lambda_k\}) = t + V_m(\{\lambda_k\})x, \tag{4.19}$$

---

[2] Formula (4.17) is due to T. Grava (2001, private communication); similar formula for the case of Whitham equations was obtained by Krichever (1989b).

defines implicit solution $\{\lambda_m(x, t)\}$ *of the system of hydrodynamic type (4.13),* *where* $\Phi_m(\lambda_1, ..., \lambda_M)$ *is an arbitrary solution of the system of differential* *equations*

$$\frac{\partial_n \Phi_m}{\Phi_m - \Phi_n} = \frac{\partial_n V_m}{V_m - V_n},\tag{4.20}$$

for $m, n = 1, ..., M$.

To apply the hodograph method to any of these systems, we need to solve also the system of equations (4.20) for functions Φ_m. Obviously, the solution is given by same formulae as the solution of equations (4.14) for functions V_m, but with another arbitrary Hölder-continuous function $h_2(P)$ independent of $\{\lambda_m\}$,

$$\Phi_m = \frac{\oint_l h_2(P) B_m(P)}{\oint_l h(P) B_m(P)}.\tag{4.21}$$

For each choice of $h_2(P)$, the system of equations (4.19) defines the implicit solution $\{\lambda_m(x, t)\}$ of the system of hydrodynamic type (4.13).

We proved the following

Theorem 4.3. *Consider the system of hydrodynamic type (4.13), where* *velocities* V_m *are given by formula (4.17) with arbitrary Hölder-continuous* *function and independent of* $\{\lambda_m\}$ *functions* $h(P)$ *and* $h_1(P)$ *on contour l. Let* $h_2(P)$ *be another arbitrary and independent of* $\{\lambda_m\}$ *Hölder-continuous function* *on contour l. Then system of M equations for M variables* $\{\lambda_m(x, t)\}_{m=1}^{M}$

$$\oint_l \{h_2(P) + h(P)t + h_1(P)x\} B_m(P) = 0, \quad m = 1, ..., M,\tag{4.22}$$

defines implicit solution $\{\lambda_m(x, t)\}$ *of the system (4.13).*

As before, the condition of Hölder continuity of functions h, h_1 and h_2 on contour l can be relaxed. Namely, we can substitute the contour l by an arbitrary subset of $\mathcal{L}$, and define on this subset three arbitrary measures independent of $\{\lambda_m\}$; the only requirement one needs to impose is commutativity of integration in (4.22) with differentiation with respect to $\{\lambda_m\}$.

5. Summary and outlook

In this paper, we propose a new class of integrable systems of partial differential equations associated to spaces of generic rational maps of fixed degree. For maps of degree two, such systems give rise to the Ernst equation from general relativity; for maps of higher degree, our systems realize the scheme of deformation of autonomous integrable systems proposed by Burtsev, Mikhailov and Zakharov. We introduce the notion of the tau-function of the new systems, and describe their relationship to the matrix Riemann–Hilbert problem and the Schlesinger system.

We generalize our construction to derive integrable systems associated to arbitrary Hurwitz spaces $H_{g,N}$ of meromorphic functions of degree N on Riemann surfaces of genus $g \geq 2$.

When the matrix dimension equals 1, our systems are linear; they can be solved via scalar Riemann–Hilbert problem on the Riemann surfaces. Each solution of such system corresponds to a flat diagonal metric (Darboux–Egoroff metric), together with corresponding systems of hydrodynamic type and their solutions.

Our results suggest the following directions of future work. We expect our systems for $g \geq 2$ to be natural deformations of two-dimensional version of Hitchin systems proposed in Krichever (2002b). The isomonodromic deformations in higher genus briefly discussed here should be in close relation to existing frameworks (Korotkin & Samtleben 1997; Levin & Olshanetski 1997; Krichever 2002a), as well as to non-autonomous Hitchin systems (Hitchin 1987). All these links should be clarified. The applications of the new systems should also be studied, especially from the point of view of their potential relationship with structures of Frobenius type.

Remark 5.1. This work was done in 2001; since then several further developments along the lines discussed here have taken place. In Kokotov & Korotkin (2004), using the relation (1.12) between rotation coefficients of flat metrics corresponding to Hurwitz Frobenius manifolds (Dubrovin 1996) and canonical meromorphic bidifferential on Riemann surfaces, it was shown that the isomonodromic tau-function of the Hurwitz Frobenius manifolds is closely related to the isomonodromic tau-function of matrix Riemann–Hilbert problems with quasi-permutation monodromy matrices (Korotkin 2004; Kokotov & Korotkin 2006). The canonical meromorphic bidifferential turned out to be a very useful tool in description of Dubrovin's Hurwitz Frobenius manifolds. Generalization of this formalism by Shramchenko (2005a,b) led to constructions of new classes (called 'deformations' and 'real doubles') of Frobenius manifolds associated to Hurwitz spaces.

This paper has also some overlap with the paper by Krichever (2002a), devoted to isomonodromy deformations in higher genus, which appeared simultaneously with the first version of this text (Kokotov & Korotkin 2001).

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