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DETERMINANTS OF PSEUDO-LAPLACIANS

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Abstract. We derive comparison formulas relating the zeta-regularized determinant of an arbitrary self-adjoint extension of the Laplace operator with domain $C_c^\infty(X\setminus\{P\})\subset L_2(X)$ to the zeta-regularized determinant of the Laplace operator on X. Here, X is a compact Riemannian manifold of dimension 2 or 3; $P\in X$.

1. Introduction

Let X_d be a complete Riemannian manifold of dimension $d\geq 2$ and let Δ be the (positive) Laplace operator on X_d. Choose a point $P\in X_d$ and consider Δ as an unbounded symmetric operator in the space $L_2(X_d)$ with domain $C_c^\infty(X_d\setminus\{P\})$. It is well known that thus obtained operator is essentially self-adjoint if and only if $d\geq 4$. In case $d=2,3$, it has deficiency indices $(1,1)$ and there exists a one-parameter family $\Delta_{\alpha,P}$ of its self-adjoint extensions (called pseudo-laplacians; see [3]). One of these extensions (the Friedrichs extension $\Delta_{0,P}$) coincides with the self-adjoint operator Δ on X_d. In case $X_d=R^d$, $d=2,3$ the scattering theory for the pair $(\Delta_{\alpha,P},\Delta)$ was extensively studied in the literature (see, e.g., [1]). The spectral theory of the operator $\Delta_{\alpha,P}$ on compact manifolds X_d ($d=2,3$) was studied in [3], notice also a recent paper [15] devoted to the case, where X_d is a compact Riemann surface equipped with Poincaré metric.

The zeta-regularized determinant of Laplacian on a compact Riemannian manifold was introduced in [11] and since then was studied and used in an immense number of papers in string theory and geometric analysis, for our future purposes we mention here the memoir [5], where the determinant of Laplacian is studied as a functional on the space of smooth Riemannian metrics on a compact two-dimensional manifold, and the papers [6, 13], where the reader may find explicit calculation of the determinant of Laplacian for three-dimensional flat tori and for the sphere S^3 (respectively).

The main result of the present paper is a comparison formula relating $\det(\Delta_{\alpha,P}-\lambda)$ to $\det(\Delta-\lambda)$, for $\lambda\in\mathbb{C}\setminus(\mathrm{Spectrum}(\Delta)\cup\mathrm{Spectrum}(\Delta_{\alpha,P}))$ (see Theorem 1 in Section 4 and Theorem 2 in Section 5).

It should be mentioned that in case of two-dimensional manifolds, the zeta regularization of $\det(\Delta_{\alpha,P}-\lambda)$ is not that standard, since the corresponding operator zeta-function has logarithmic singularity at 0.

It should be also mentioned that in the case when the manifold X_d is flat in a vicinity of the point P we deal with a very special case of the situation (Laplacian on a manifold with conical singularity) considered in [8–10] and, via other method, in [7]. The general scheme of the present work is close to that of [7], although some calculations from [9] also appear very useful for us.

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2. Pseudo-Laplacians, Krein formula and scattering coefficient

Let X_d be a compact Riemannian manifold of dimension $d = 2$ or $d = 3$; $P \in X_d$ and $\alpha \in [0, \pi)$. Following Colin de Verdière [3], introduce the set

$$\mathcal{D}(\Delta_{\alpha,P}) = \{f \in H^2(X_d \setminus \{P\}) : \exists c \in \mathbb{C} : \textit{ in a vicinity of } P \textit{ one has}$$
$$f(x) = c(\sin\alpha \cdot G_d(r) + \cos\alpha) + o(1) \textit{ as } r \to 0\}, \tag{2.1}$$

where

$$H^2(X_d \setminus \{P\}) = \{f \in L_2(X_d) : \exists C \in \mathbb{C} : \Delta f - C\delta_P \in L_2(X_d)\},$$

r is the geodesic distance between x and P and

$$G_d(r) = \begin{cases} \frac{1}{2\pi}\log r, & d = 2, \\ -\frac{1}{4\pi r}, & d = 3. \end{cases}$$

Then (see [3]) the self-adjoint extensions of symmetric operator Δ with domain $C_c^\infty(X_d \setminus \{P\})$ are the operators $\Delta_{\alpha,P}$ with domains $\mathcal{D}(\Delta_{\alpha,P})$ acting via $u \mapsto \Delta u$. The extension $\Delta_{0,P}$ coincides with the Friedrichs extension and is nothing but the self-adjoint Laplacian on X_d.

Let $R(x, y; \lambda)$ be the resolvent kernel of the self-adjoint Laplacian Δ on X_d.

Following [3] define the scattering coefficient $F(\lambda; P)$ via

$$-R(x, P; \lambda) = G_d(r) + F(\lambda; P) + o(1) \tag{2.2}$$

as $x \to P$. (Note that in [3] the resolvent is defined as $(\lambda - \Delta)^{-1}$, whereas for us it is $(\Delta - \lambda)^{-1}$. This results in the minus sign in (2.2).)

As it was already mentioned the deficiency indices of the symmetric operator Δ with domain $C_c^\infty(X_d \setminus \{P\})$ are $(1, 1)$, therefore, one has the following Krein formula (see, e.g., [1], p. 357) for the resolvent kernel, $R_\alpha(x, y; \lambda)$, of the self-adjoint extension $\Delta_{\alpha,P}$:

$$R_\alpha(x, y; \lambda) = R(x, y; \lambda) + k(\lambda; P)R(x, P; \lambda)R(P, y; \lambda) \tag{2.3}$$

with some $k(\lambda; P) \in \mathbb{C}$.

The following Lemma relates $k(\lambda; P)$ to the scattering coefficient $F(\lambda; P)$.

Lemma 1. *One has the relation*

$$k(\lambda; P) = \frac{\sin\alpha}{F(\lambda; P)\sin\alpha - \cos\alpha}. \tag{2.4}$$

Proof. Send $x \to P$ in (2.3), observing that $R_\alpha(\,\cdot\,, y; \lambda)$ belongs to $\mathcal{D}(\Delta_{\alpha,P})$, make use of (2.1) and (2.2), and then compare the coefficients near $G_d(r)$ and the constant terms in the asymptotical expansions at the left and at the right. □

It follows in particular from the Krein formula that the difference of the resolvents $(\Delta_{\alpha,P} - \lambda)^{-1} - (\Delta - \lambda)^{-1}$ is a rank one operator. The following simple Lemma is the key observation of the present work.

Lemma 2. *One has the relation*

$$\operatorname{Tr}\left((\Delta_{\alpha,P} - \lambda)^{-1} - (\Delta - \lambda)^{-1}\right) = \frac{F'_\lambda(\lambda; P)\sin\alpha}{\cos\alpha - F(\lambda; P)\sin\alpha}. \tag{2.5}$$

Proof. One has

$$-F'_\lambda(\lambda;P)=\left.\frac{\partial R(y,P;\lambda)}{\partial\lambda}\right|_{y=P}=\lim_{\mu\to\lambda}\frac{R(y,P;\mu)-R(y,P;\lambda)}{\mu-\lambda}.$$

Using resolvent identity, we rewrite the last expression as

$$\lim_{\mu\to\lambda}\int_{X_d}R(y,z;\mu)R(P,z;\lambda)dz\bigg|_{y=P}=\int_{X_d}[R(P,z;\lambda)]^2dz.$$

From (2.3) it follows that

$$[R(P,z;\lambda)]^2=\frac{1}{k(\lambda;P)}\left(R_{\alpha,P}(x,z;\lambda)-R(x,z;\lambda)\right)\bigg|_{x=z}.$$

This implies

$$-F'_\lambda(\lambda;P)=\frac{1}{k(\lambda,P)}\mathrm{Tr}\left((\Delta_{\alpha,P}-\lambda)^{-1}-(\Delta-\lambda)^{-1}\right),$$

which, together with Lemma 1, imply (2.5). □

Introduce the domain

$$\Omega_{\alpha,P}=\mathbb{C}\setminus\{\lambda-it,\lambda\in\mathrm{Spectrum}\,(\Delta)\cup\mathrm{Spectrum}\,(\Delta_{\alpha,P});t\in(-\infty,0]\}.$$

Then in $\Omega_{\alpha,P}$ one can introduce the function

$$\tilde{\xi}(\lambda)=-\frac{1}{2\pi i}\log(\cos\alpha-F(\lambda;P)\sin\alpha) \tag{2.6}$$

(It should be noted that the function $\xi=\Re(\tilde{\xi})$ is the spectral shift function of Δ and $\Delta_{\alpha,P}$.) One can rewrite (2.5) as

$$\mathrm{Tr}\left((\Delta_{\alpha,P}-\lambda)^{-1}-(\Delta-\lambda)^{-1}\right)=2\pi i\tilde{\xi}'(\lambda). \tag{2.7}$$

3. Operator zeta-function of $\boldsymbol{\Delta_{\alpha,P}}$

Denote by $\zeta(s,A)$ the zeta-function

$$\zeta(s,A)=\sum_{\mu_k\in\mathrm{Spectrum}\,(A)}\frac{1}{\mu_k^s}$$

of the operator A. (We assume that the spectrum of A is discrete and does not contain 0.)

Take any $\tilde{\lambda}$ from $\mathbb{C}\setminus(\mathrm{Spectrum}\,(\Delta_{\alpha,P})\cup\mathrm{Spectrum}\,(\Delta)))$. From the results of [3] it follows that the function $\zeta(s,\Delta_{\alpha,P}-\tilde{\lambda})$ is defined for sufficiently large $\Re s$. It is well known that $\zeta(s,\Delta-\tilde{\lambda})$ is meromorphic in $\mathbb{C}$.

The proof of the following lemma coincides verbatim with the proof of Proposition 5.9 from [7].

Lemma 3. *Suppose that the function $\tilde{\xi}'(\lambda)$ from (2.7) is $O\left(|\lambda|^{-1}\right)$ as $\lambda\to-\infty$. Let $-C$ be a sufficiently large negative number and let $c_{\tilde{\lambda},\epsilon}$ be a contour encircling the* *cut* $c_{\tilde\lambda}$ *which starts from* $-\infty+0i$*, follows the real line till* $-C$ *and then goes to* $\tilde\lambda$ *remaining in* $\Omega_{\alpha,P}$*. Assume that* $\operatorname{dist}(z, c_{\tilde\lambda})=\epsilon$ *for any* $z\in c_{\tilde\lambda,\epsilon}$*. Let also*

$$\zeta_2(s)=\int_{c_{\tilde\lambda,\epsilon;2}}(\lambda-\tilde\lambda)^{-s}\tilde\xi'(\lambda)d\lambda,$$

where the the integral at the right-hand side is taken over the part $c_{\tilde\lambda,\epsilon;2}$ *of the contour* $c_{\tilde\lambda,\epsilon}$ *lying in the half-plane* $\{\lambda:\Re\lambda>-C\}$*. Let*

$$\hat\zeta_2(s)=\lim_{\epsilon\to0}\zeta_2(s)=2i\sin(\pi s)\int_{-C}^{\tilde\lambda}(\lambda-\tilde\lambda)_0^{-s}\tilde\xi'(\lambda)\,d\lambda,$$

where $(\lambda-\tilde\lambda)_0^{-s}=e^{-i\pi s}\lim_{\lambda\downarrow c_{\tilde\lambda}}(\lambda-\tilde\lambda)^{-s}$*. Then the function*

$$R(s,\tilde\lambda)=\zeta(s,\Delta_{\alpha,P}-\tilde\lambda))-\zeta(s,\Delta-\tilde\lambda)-2i\sin(\pi s)\int_{-\infty}^{-C}|\lambda|^{-s}\tilde\xi'(\lambda)d\lambda-\hat\zeta_2(s) \tag{3.1}$$

can be analytically continued to $\Re s>-1$ *with* $R(0,\tilde\lambda)=R'_s(0,\tilde\lambda)=0$.

For completeness, we give a sketch of proof.

Proof. Using (2.7), one has for sufficiently large $\Re s$

$$\zeta(s,\Delta_{\alpha,P}-\tilde\lambda)-\zeta(s,\Delta-\tilde\lambda)=\frac{1}{2\pi i}\int_{c_{\tilde\lambda,\epsilon}}(\lambda-\tilde\lambda)^{-s}\mathrm{Tr}((\Delta_{\alpha,P}-\lambda)^{-1}-(\Delta-\lambda)^{-1})d\lambda$$
$$=\int_{c_{\tilde\lambda,\epsilon}}(\lambda-\tilde\lambda)^{-s}\tilde\xi'(\lambda)\,d\lambda=\zeta_1(s)+\zeta_2(s),$$

where

$$\zeta_1(s)=\left\{\int_{-\infty+i\epsilon}^{-C+i\epsilon}-\int_{-\infty-i\epsilon}^{-C-i\epsilon}\right\}(\lambda-\tilde\lambda)^{-s}\tilde\xi'(\lambda)d\lambda.$$

It is easy to show (see Lemma 5.8 in [7]) that in the limit $\epsilon\to0$ $\zeta_1(s)$ gives

$$2i\sin(\pi s)\int_{-\infty}^{-C}|\lambda|^{-s}\tilde\xi'(\lambda)\,d\lambda+2i\sin(\pi s)\int_{-\infty}^{-C}|\lambda|^{-s}\tilde\xi'(\lambda)\rho(s,\tilde\lambda/\lambda)d\lambda, \tag{3.2}$$

where $\rho(s,z)=(1+z)^{-s}-1$ and

$$\rho(s,\tilde\lambda/\lambda)=O\left(|\lambda|^{-1}\right),$$

as $\lambda\to-\infty$. Using the assumption on the asymptotics of $\tilde\xi(\lambda)$ as $\lambda\to-\infty$ and the obvious relation $\rho(0,z)=0$, one can see that the last term in (3.2) can be analytically continued to $\Re s>-1$ and vanishes together with its first derivative with respect to s at $s=0$. Denoting it by $R(s,\tilde\lambda)$, one gets the Lemma. □

As it is stated in the introduction the main object, we are to study in the present paper is the zeta-regularized determinant of the operator $\Delta_{\alpha,P}-\lambda$. Let us remind the reader that the usual definition of the zeta-regularized determinant of an operator A

$$\det A=\exp\left(-\zeta'(0,A)\right) \tag{3.3}$$

requires analyticity of $\zeta(s,A)$ at $s=0$.

Since the operator zeta-function $\zeta(s,\Delta-\tilde\lambda)$ is regular at $s=0$ (in fact, it is true in case of Δ being an arbitrary elliptic differential operator on any compact manifold) and the function $\hat{\zeta}_2(s)$ is entire, Lemma 3 shows that the behavior of the function $\zeta(s, \Delta_{\alpha,P} - \tilde{\lambda})$ at $s = 0$ is determined by the properties of the analytic continuation of the term

$$2i \sin(\pi s) \int_{-\infty}^{-C} |\lambda|^{-s} \tilde{\xi}'(\lambda) d\lambda \tag{3.4}$$

in (3.1). These properties in their turn are determined by the asymptotical behavior of the function $\tilde{\xi}'(\lambda)$ as $\lambda \to -\infty$.

It turns out that the latter behavior depends on dimension d. In particular, in the next section, we will find out that in case $d = 2$ the function $\zeta(s, \Delta_{\alpha,P} - \tilde{\lambda})$ is not regular at $s = 0$; therefore, in order to define $\det(\Delta_{\alpha,P} - \tilde{\lambda})$ one has to use a modified version of (3.3).

4. Determinant of pseudo-Laplacian on two-dimensional compact manifold

Let X be a two-dimensional Riemannian manifold, then introducing isothermal local coordinates (x, y) and setting $z = x + iy$, one can write the area element on X as

$$\rho^{-2}(z)|dz|^2.$$

The following estimate of the resolvent kernel, $R(z', z; \lambda)$, of the Laplacian on X was found by Fay (see [5]; Theorem 2.7 on page 38 and the formula preceding Corollary 2.8 on page 39; note that Fay works with negative Laplacian, so one has to take care of signs when using his formulas).

Lemma 4 (**J. Fay**). *The following equality holds true*

$$\begin{aligned} -R(z, z'; \lambda)) = G_2(r) + O(r) + \frac{1}{2\pi} \Bigg[\gamma + \log \frac{\sqrt{|\lambda| + 1}}{2} \\ - \frac{1}{2(|\lambda| + 1)} \left(1 + \frac{4}{3} \rho^2(z) \partial^2_{z\bar{z}} \log \rho(z) \right) + \hat{R}(z', z; \lambda) \Bigg], \end{aligned} \tag{4.1}$$

where $O(r)$ is λ-independent, $\hat{R}(z', z; \lambda)$ is continuous for z' near z,

$$\hat{R}(z, z; \lambda) = O\left(|\lambda|^{-2}\right)$$

uniformly with respect to $z \in X$ as $\lambda \to -\infty$; $r = \mathrm{dist}(z, z')$, γ is the Euler constant.

Using (4.1), we immediately get the following asymptotics of the scattering coefficient $F(\lambda, P)$ as $\lambda \to -\infty$:

$$\begin{aligned} F(\lambda, P) = \frac{1}{4\pi} \log(|\lambda| + 1) + \frac{\gamma - \log 2}{2\pi} \\ - \frac{1}{4\pi(|\lambda| + 1)} \left[1 + \frac{4}{3} \rho^2(z) \partial^2_{z\bar{z}} \log \rho(z) \Big|_{z = z(P)} \right] + O\left(|\lambda|^{-2}\right). \end{aligned} \tag{4.2}$$

Remark 1. It is obvious that the expression $\rho^2(z) \partial^2_{z\bar{z}} \log \rho(z) \big|_{z = z(P)}$ is independent of the choice of conformal local parameter z near P.

Now from (2.6) and (4.2) it follows that

$$2\pi i\tilde{\xi}'(\lambda) = -\frac{\frac{1}{4\pi(|\lambda|+1)} - \frac{b}{(|\lambda|+1)^2} + O\left(|\lambda|^{-3}\right)}{\cot\alpha - a - \frac{1}{4\pi}\log(|\lambda|+1) + \frac{b}{|\lambda|+1} + O\left(|\lambda|^{-2}\right)},$$

where $a = \frac{1}{2\pi}(\gamma - \log 2)$ and $b = \frac{1}{4\pi}(1 + \frac{4}{3}\rho^2\partial^2_{z\bar{z}}\log\rho)$. This implies that for $-\infty < \lambda \leq -C$, one has

$$2\pi i\tilde{\xi}'(\lambda) = \frac{1}{|\lambda|(\log|\lambda| - 4\pi\cot\alpha + 4\pi a)} + f(\lambda), \tag{4.3}$$

with $f(\lambda) = O\left(|\lambda|^{-2}\right)$ as $\lambda \to -\infty$. Now knowing (4.3), one can study the behavior of the term (3.4) in (3.1). We have

$$2i\sin(\pi s)\int_{-\infty}^{-C}|\lambda|^{-s}\tilde{\xi}'(\lambda)d\lambda = \frac{\sin(\pi s)}{\pi}\int_{-\infty}^{-C}|\lambda|^{-s-1}\frac{d\lambda}{(\log|\lambda| - 4\pi\cot\alpha + 4\pi a)} + \frac{\sin(\pi s)}{\pi}\int_{-\infty}^{-C}|\lambda|^{-s}f(\lambda)\,d\lambda. \tag{4.4}$$

The first integral in the right hand side of (4.4) appeared in ([9], p. 15), where it was observed that it can be easily rewritten through the function

$$\mathrm{Ei}(z) = -\int_{-z}^{\infty} e^{-y}\frac{dy}{y} = \gamma + \log(-z) + \sum_{k=1}^{\infty}\frac{z^k}{k\cdot k!},$$

which leads to the representation

$$\frac{\sin(\pi s)}{\pi}\int_{-\infty}^{-C}|\lambda|^{-s-1}\frac{d\lambda}{(\log|\lambda| - 4\pi\cot\alpha + 4\pi a)} = -\frac{\sin(\pi s)}{\pi}e^{-s\kappa}\left[\gamma + \log(s(\log C - \kappa)) + e(s)\right], \tag{4.5}$$

where $e(s)$ is an entire function such that $e(0) = 0$; $\kappa = 4\pi\cot\alpha - 4\pi a$. From this we conclude that

$$\frac{\sin(\pi s)}{\pi}\int_{-\infty}^{-C}|\lambda|^{-s-1}\frac{d\lambda}{(\log|\lambda| - 4\pi\cot\alpha + 4\pi a)} = -s\log s + g(s), \tag{4.6}$$

where $g(s)$ is differentiable at $s = 0$.

Now (3.1) and (4.6) justify the following definition.

Definition 1. Let $\Delta_{\alpha,P}$ be the pseudo-Laplacian on a two-dimensional compact Riemannian manifold. Then the zeta-regularized determinant of the operator $\Delta_{\alpha,P} - \tilde{\lambda}$ with $\tilde{\lambda} \in \mathbb{C} \setminus \mathrm{Spectrum}(\Delta_{\alpha,P})$ is defined as

$$\det(\Delta_{\alpha,P} - \tilde{\lambda}) = \exp\left\{-\frac{d}{ds}\left[\zeta(s, \Delta_{\alpha,P} - \tilde{\lambda}) + s\log s\right]\Big|_{s=0}\right\}. \tag{4.7}$$

We are ready to get our main result: the formula relating $\det(\Delta_{\alpha,P} - \tilde{\lambda})$ to $\det(\Delta - \tilde{\lambda})$.

From (3.1), (3.4) it follows that

$$\begin{aligned}(4.8)\qquad &\frac{d}{ds}\left[\zeta(s,\Delta_{\alpha,P}-\tilde\lambda)+s\log s-\zeta(s,\Delta-\tilde\lambda)\right]\Big|_{s=0}\\ &=\frac{d}{ds}\hat\zeta_2(s)\Big|_{s=0}+\int_{-\infty}^{-C}f(\lambda)\,d\lambda\\ &\quad-\frac{d}{ds}\left\{\frac{\sin\pi s}{\pi}e^{-s\kappa}\left[\gamma+\log(s(\log C-\kappa))+e(s)\right]+s\log s\right\}\Big|_{s=0}\\ &=2\pi i\left(\tilde\xi(\tilde\lambda)-\tilde\xi(-C)\right)+\int_{-\infty}^{-C}f(\lambda)\,d\lambda-\gamma-\log(\log C-\kappa)\\ &=\times 2\pi i\tilde\xi(\tilde\lambda)-\gamma\\ &\quad+\int_{-\infty}^{-C}f(\lambda)\,d\lambda-2\pi i\tilde\xi(-C)-\log(\log C-4\pi\cot\alpha+2\gamma-\log 4).\end{aligned}$$

Note that the expression in the second line of (4.8) should not depend on C, so one can send C to $+\infty$ there. Together with (4.2) this gives

$$\frac{d}{ds}\left[\zeta(s,\Delta_{\alpha,P}-\tilde\lambda)+s\log s-\zeta(s,\Delta-\tilde\lambda)\right]\Big|_{s=0}=2\pi i\tilde\xi(\tilde\lambda)-\gamma+\log(\sin\alpha/(4\pi))-i\pi,$$

which implies the comparison formula for the determinants stated in the following theorem.

Theorem 1. *Let $d=2$, suppose $\tilde\lambda$ does not belong to the union of spectra of Δ and $\Delta_{\alpha,P}$ and let the zeta-regularized determinant of $\Delta_{\alpha,P}$ be defined as in (4.7). Then one has the relation*

$$\det(\Delta_{\alpha,P}-\tilde\lambda)=-4\pi e^{\gamma}(\cot\alpha-F(\tilde\lambda,P))\det(\Delta-\tilde\lambda). \tag{4.9}$$

Observe now that 0 is the simple eigenvalue of Δ and, therefore, it follows from Theorem 2 in [3] that 0 does not belong to the spectrum of the operator $\Delta_{\alpha,P}$ and that $\Delta_{\alpha,P}$ has one strictly negative simple eigenvalue when $\alpha\neq 0$. Thus, the determinant in the left-hand side of (4.9) is well defined for $\tilde\lambda=0$, whereas the determinant at the right-hand side has the asymptotics

$$\det(\Delta-\tilde\lambda)\sim(-\tilde\lambda){\det}^*\Delta \tag{4.10}$$

as $\tilde\lambda\to 0-$. Here ${\det}^*\Delta$ is the modified determinant of an operator with zero mode.

From the standard asymptotics

$$-R(x,y;\lambda)=\frac{1}{\mathrm{Vol}(X)}\frac{1}{\lambda}+G_2(r)+O(1)$$

as $\lambda\to 0$ and $x\to y$ one gets the asymptotics

$$F(\lambda,P)=\frac{1}{\mathrm{Vol}(X)}\frac{1}{\lambda}+O(1) \tag{4.11}$$

as $\lambda\to 0$. Now sending $\tilde\lambda\to 0-$ in (4.9) and using (4.10) and (4.11) we get the following corollary of the Theorem 1.

Corollary 1. *For $\alpha \in (0, \pi)$ the following relation holds true:*

$$\det\Delta_{\alpha,P} = -\frac{4\pi e^{\gamma}}{\mathrm{Vol}(X)}\det{}^*\Delta. \tag{4.12}$$

5. Determinant of pseudo-Laplacian on three-dimensional manifolds

Let X be a three-dimensional compact Riemannian manifold. We start with the Lemma describing the asymptotical behavior of the scattering coefficient as $\lambda \to -\infty$.

Lemma 5. *One has the asymptotics*

$$F(\lambda; P) = \frac{1}{4\pi}\sqrt{-\lambda} + c_1(P)\frac{1}{\sqrt{-\lambda}} + O\left(|\lambda|^{-1}\right), \tag{5.1}$$

as $\lambda \to -\infty$

Proof. Consider Minakshisundaram–Pleijel asymptotic expansion [12]

$$H(x, P; t) = (4\pi t)^{-3/2}\, e^{-d(x,P)^2/(4t)} \sum_{k=0}^{\infty} u_k(x, P)t^k \tag{5.2}$$

for the heat kernel in a small vicinity of P, here $d(x, P)$ is the geodesic distance from x to P, functions $u_k(\cdot, P)$ are smooth in a vicinity of P, the equality is understood in the sense of asymptotic expansions. We will make use of the standard relation

$$R(x, y; \lambda) = \int_0^{+\infty} H(x, y; t)e^{\lambda t}\, dt. \tag{5.3}$$

Let us first truncate the sum (5.2) at some fixed $k = N + 1$, so that the remainder, r_n, is $O\left(t^N\right)$. Defining

$$\tilde{R}_N(x, P; -\lambda) := \int_0^{\infty} r_n(t, x, P)e^{t\lambda} dt,$$

we see that

$$\tilde{R}_N(x, P; \lambda) = O\left(|\lambda|^{-(N+1)}\right)$$

as $\lambda \to -\infty$ uniformly with respect to x belonging to a small vicinity of P.

Now, for each $0 \le k \le N + 1$ we have to address the following quantity

$$R_k(x, P; \lambda) := \frac{u_k(x, y)}{(4\pi)^{3/2}} \int_0^{\infty} t^{k-\frac{3}{2}} e^{-\frac{d(x,P)^2}{4t}} e^{\lambda t} dt.$$

According to identity (5.12) below, one has

$$R_0(x, P; \lambda) = \frac{u_0(x, P)}{(4\pi)^{3/2}}\frac{2\sqrt{\pi}}{d(x, P)} e^{-d(x,P)\sqrt{-\lambda}} = \frac{1}{4\pi d(x, P)} - \frac{1}{4\pi}\sqrt{-\lambda} + o(1), \tag{5.4}$$

as $d(x, P) \to 0$. For $k \ge 1$ one has

$$\begin{aligned} R_k(x, P; \lambda) &= \frac{u_k(x, P)}{(4\pi)^{3/2}} 2^{3/2-k} \left(\frac{d(x, P)}{\sqrt{-\lambda}}\right)^{k-1/2} K_{k-\frac{1}{2}}(d(x, P)\sqrt{-\lambda}) \\ &= -c_k(P)\frac{1}{(\sqrt{-\lambda})^{2k-1}} + o(1) \end{aligned} \tag{5.5}$$

as $d(x, P) \to 0$ (see [2], p. 146, f-la 29). Now (5.1) follows from (5.3) to (5.5). □

Now from Lemma 5 it follows that

$$2\pi i\tilde{\xi}'(\lambda) = -\frac{1}{2\lambda} + O\left(|\lambda|^{-3/2}\right) \tag{5.6}$$

as $\lambda \to -\infty$, therefore, one can rewrite (3.4) as

$$\frac{\sin(\pi s)}{\pi}\left\{\int_{-\infty}^{-C} |\lambda|^{-s}\left(2\pi i\tilde{\xi}'(\lambda) + \frac{1}{2\lambda}\right) d\lambda + \frac{C^{-s}}{2s}\right\}, \tag{5.7}$$

which is obviously analytic in $\Re s > -\frac{1}{2}$. Thus, it follows from (3.1) that the function $\zeta(s, \Delta_{\alpha,P} - \tilde{\lambda})$ is regular at $s = 0$ and one can introduce the usual zeta-regularization

$$\det(\Delta_{\alpha,P} - \tilde{\lambda}) = \exp\{-\zeta'(0, \Delta_{\alpha,P} - \tilde{\lambda})\}$$

of $\det(\Delta_{\alpha,P} - \tilde{\lambda})$.

Moreover, differentiating (3.1) with respect to s at $s = 0$ similarly to (4.8) we get

$$\begin{aligned} &\frac{d}{ds}\left[\zeta(s, \Delta_{\alpha,P} - \tilde{\lambda}) - \zeta(s, \Delta - \tilde{\lambda})\right]\Big|_{s=0} \\ &= 2\pi i(\tilde{\xi}(\tilde{\lambda}) - \tilde{\xi}(-C)) + \int_{-\infty}^{-C}\left(2\pi i\tilde{\xi}'(\lambda) + \frac{1}{2\lambda}\right) d\lambda - \frac{1}{2}\log C, \end{aligned}$$

which reduces after sending $-C \to -\infty$ to

$$2\pi i\tilde{\xi}(\tilde{\lambda}) + \log\sin\alpha - \log(4\pi) + i\pi = -\log(\cot\alpha - F(\lambda; P)) - \log(4\pi) + i\pi,$$

which implies the following theorem.

Theorem 2. *For $d = 3$ let $\Delta_{\alpha,P}$ be the pseudo-Laplacian on X and $\tilde{\lambda} \in \mathbb{C} \setminus (\mathrm{Spectrum}(\Delta) \cup \mathrm{Spectrum}(\Delta_{\alpha,P}))$. Then*

$$\det(\Delta_{\alpha,P} - \tilde{\lambda}) = -4\pi(\cot\alpha - F(\tilde{\lambda}; P))\det(\Delta - \tilde{\lambda}). \tag{5.8}$$

Sending $\tilde{\lambda} \to 0$ and noting that relation (4.11) holds also in case $d = 3$, we get the following corollary.

Corollary 2. *For $\alpha \in (0, \pi)$*

$$\det\Delta_{\alpha,P} = -\frac{4\pi}{\mathrm{Vol}(X)}\det{}^*\Delta. \tag{5.9}$$

In what follows, we consider two examples of three-dimensional compact Riemannian manifolds for which there exist explicit expressions for the resolvent kernels: a flat torus and the round (unit) $3d$-sphere. These manifolds are homogeneous, so, as it is shown in [3], the scattering coefficient $F(\lambda, P)$ is P-independent.

Example 1: Round $3d$-sphere.

Lemma 6. *Let $X = S^3$ with usual round metric. Then there is the following explicit expression for scattering coefficient*

$$F(\lambda) = \frac{1}{4\pi}\coth\left(\pi\sqrt{-\lambda - 1}\right) \cdot \sqrt{-\lambda - 1} \tag{5.10}$$

and, therefore, one has the following asymptotics as $\lambda \to -\infty$:

$$F(\lambda) = \frac{1}{4\pi}\sqrt{|\lambda| - 1} + O\left(|\lambda|^{-\infty}\right). \tag{5.11}$$

Remark 2. The possibility of finding an explicit expression for $F(\lambda)$ for S^3 was mentioned in [3]. However, we failed to find (5.10) in the literature.

Proof. We will make use the well-known identity (see, e.g., [2], p. 146, f-la 28):

$$\int_0^{+\infty} e^{\lambda t} t^{-3/2}\, e^{-\frac{d^2}{4t}}\, dt = 2\frac{\sqrt{\pi}}{|d|} e^{-|d|\sqrt{-\lambda}}; \tag{5.12}$$

for $\lambda < 0$ and $d \in \mathbb{R}$ and the following explicit formula for the operator kernel $e^{-t}H(x, y; t)$ of the operator $e^{-t(\Delta+1)}$, where Δ is the (positive) Laplacian on S^3 (see [4], (2.29)):

$$e^{-t}H(x, y; t) = -\frac{1}{2\pi}\frac{1}{\sin d(x, y)}\frac{\partial}{\partial z}\Big|_{z=d(x,y)}\Theta(z, t). \tag{5.13}$$

Here, $d(x, y)$ is the geodesic distance between $x, y \in S^3$ and

$$\Theta(z, t) = \frac{1}{\sqrt{4\pi t}}\sum_{k=-\infty}^{+\infty} e^{-(z+2k\pi)^2/4t}$$

is the theta-function.

Denoting $d(x, y)$ by θ and using (5.13) and (5.12), one gets

$$\begin{aligned} R(x, y; \lambda - 1) &= \int_0^{+\infty} e^{\lambda t}\, e^{-t}H(x, y; t)\, dt \\ &= \frac{1}{4\pi}\frac{1}{\sin\theta}\left(-\sum_{k<0} e^{(\theta+2k\pi)\sqrt{-\lambda}} + \sum_{k\geq 0} e^{-(\theta+2k\pi)\sqrt{-\lambda}}\right) \\ &= \frac{1}{4\pi}\frac{1}{\sin\theta}\frac{1}{1 - e^{-2\pi\sqrt{-\lambda}}}\left[-e^{-2\pi\sqrt{-\lambda}}e^{\theta\sqrt{-\lambda}} + e^{-\theta\sqrt{-\lambda}}\right] \\ &= \frac{1}{4\pi\theta} - \frac{1}{4\pi}\frac{1 + e^{-2\pi\sqrt{-\lambda}}}{1 - e^{-2\pi\sqrt{-\lambda}}}\sqrt{-\lambda} + o(1) \end{aligned} \tag{5.14}$$

as $\theta \to 0$, which implies the Lemma. □

Example 2 (flat 3d-tori). Let $\{\mathbf{A}, \mathbf{B}, \mathbf{C}\}$ be a basis of $\mathbb{R}^3$ and let T^3 be the quotient of $\mathbb{R}^3$ by the lattice $\{m\mathbf{A} + n\mathbf{B} + l\mathbf{C} : (m, n, l) \in \mathbb{Z}^3\}$ provided with the usual flat metric.

Note that the free resolvent kernel in R^3 is

$$\frac{e^{-\sqrt{-\lambda}\|x-y\|}}{4\pi\|x - y\|}$$

and, therefore,

$$R(x,y;\lambda)=\frac{e^{-\sqrt{-\lambda}\|x-y\|}}{4\pi\|x-y\|}+\frac{1}{4\pi}\sum_{(m,n,l)\in\mathbb{Z}^3\setminus(0,0,0)}\frac{e^{-\sqrt{-\lambda}\|x-y+m\mathbf{A}+n\mathbf{B}+l\mathbf{C}\|}}{\|x-y+m\mathbf{A}+n\mathbf{B}+l\mathbf{C}\|}. \tag{5.15}$$

From (5.15) it follows that

$$\begin{aligned}F(\lambda)&=\frac{1}{4\pi}\sqrt{-\lambda}-\frac{1}{4\pi}\sum_{(m,n,l)\in\mathbb{Z}^3\setminus(0,0,0)}\frac{e^{-\sqrt{-\lambda}\|m\mathbf{A}+n\mathbf{B}+l\mathbf{C}\|}}{\|m\mathbf{A}+n\mathbf{B}+l\mathbf{C}\|}\\&=\frac{1}{4\pi}\sqrt{-\lambda}+O\left(|\lambda|^{-\infty}\right)\end{aligned}$$

as $\lambda\to-\infty$.

Remark 3. It should be noted that explicit expressions for $\det^*\Delta$ in case $X=S^3$ and $X=T^3$ are given in [13] and [6].

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