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Nuclear Physics B 694 [PM] (2004) 443–472

NUCLEAR PHYSICS B

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Genus one contribution to free energy in Hermitian two-matrix model

B. Eynard [a], A. Kokotov [b], D. Korotkin [b]

[a] *Service de Physique Théorique, CEA/Saclay, Orme des Merisier, F-91191 Gif-sur-Yvette cedex, France*
[b] *Concordia University, 7141 Sherbrooke West, Montreal H4B1R6, Montrteal, PQ, Canada*

Received 8 March 2004; received in revised form 28 May 2004; accepted 11 June 2004

Abstract

We compute the genus one correction to free energy of Hermitian two-matrix model in large N limit in terms of theta-functions associated to the spectral curve. We discuss the relationship of this expression to the isomonodromic tau-function, the Bergmann tau-function on Hurwitz spaces, the G-function of Frobenius manifolds and the determinant of Laplacian in a singular metric over the spectral curve.

PACS: 02.90.+p

1. Two-matrix models: introduction

In this paper we study the partition function of the multi-cut two-matrix model [1,2]:

$$Z_N \equiv e^{-N^2 F} := \int dM_1\, dM_2\, e^{-N\,\mathrm{tr}\{V_1(M_1)+V_2(M_2)-M_1M_2\}}, \tag{1.1}$$

where the integral is taken over all independent entries of two Hermitian matrices M_1 and M_2 such that the eigenvalues of M_1 are concentrated over a finite set of intervals (cuts) with given filling fractions. According to widely accepted point of view, we understand the integral (1.1) as a formal asymptotic series with respect to powers of the matrix size N and the coefficients of the polynomial potentials V_1 and V_2. Therefore, here we do not discuss problems related to convergence of this matrix integral; moreover, being interpreted

E-mail address: korotkin@mathstat.concordia.ca (D. Korotkin).

0550-3213/$ – see front matter
doi:10.1016/j.nuclphysb.2004.06.031

in the formal sense, the model can be easily extended to matrices whose eigenvalues are concentrated on a set of contours in the complex plane.

The asymptotic series $F = \sum_{G=0}^{\infty} N^{-2G} F^G$ with respect to the powers of $1/N^2$ play an important role in physics. In particular, the coefficients F^G of these series can be interpreted as generating functions of statistical physics models on random discretized polygonal surfaces of genus G, which are used as simplified models of euclidean 2D quantum gravity [3,4,6]; thus an expansion of this kind is called "topological expansion". Double scaling limits of these models correspond to statistical physics models on continuous surfaces, with conformal invariance properties. According to this philosophy, the Hermitian one-matrix model corresponds to pure gravity (i.e., $q = 2$), while the Hermitian two-matrix models correspond to all (p, q) minimal models.

The interest to large N matrix models was renewed after recent discovery of a close relationship between the large N expansion of the free energy of matrix models and the low-energy effective action of some string theories [27].

The computation of $1/N^2$ expansion for both one-matrix and two-matrix models is based on the loop equations, which were first derived for the one-cut solution of the Hermitian one-matrix model in [8]. For the two-cut case of the one-matrix model, when the spectral curve has genus one, the loop equations were derived in the works [9,10], where F^1 was also found. The large N expansion for the one-matrix model in the multi-cut case was discussed in recent papers [11–13], where F^1 was computed in terms of holomorphic objects associated to the hyperelliptic spectral curve.

The loop equations for the Hermitian two-matrix model were derived in works [14,15] of one of the authors of this paper; in these works the genus one correction F^1 to the free energy was computed for the spectral curves of genus zero ("one-cut" case) and one ("two-cut" case).

In this paper we extend the results of the works [14,15] to "multi-cut" case, when the genus of the spectral curve is arbitrary (up to the maximal genus, which can be computed in terms of degrees of polynomials V_1 and V_2).

Let us write down the polynomials V_1 and V_2 in the form

$$V_1(x) = \sum_{k=1}^{d_1+1} \frac{u_k}{k} x^k, \qquad V_2(y) = \sum_{k=1}^{d_2+1} \frac{v_k}{k} y^k. \tag{1.2}$$

We shall use the following standard notations for the operators of differentiation with respect to coefficients of these polynomials:

$$\frac{\delta}{\delta V_1(x)}\bigg|_x := \sum_{k=1}^{d_1+1} x^{-k-1} k \partial_{u_k}, \qquad \frac{\delta}{\delta V_2(y)}\bigg|_y := \sum_{k=1}^{d_2+1} y^{-k-1} k \partial_{v_k}. \tag{1.3}$$

This notation will be used below to shorten some of the formulas; by definition the equality

$$\frac{\delta F}{\delta V_1(x)}\bigg|_x = H(x)$$

means that

$$\frac{\partial F}{\partial u_k} = \frac{1}{2\pi i k} \oint_{x=\infty} x^k H(x)\, dx, \quad k = 1, \ldots, d_1 + 1; \tag{1.4}$$

a detailed discussion of this notation is contained in [17]. In fact, formally it is much more convenient not to cut the functions V_1 and V_1 to polynomials, but to consider instead the Laurent series

$$V_1(x) = \sum_{k=1}^{\infty} \frac{u_k}{k} x^k, \qquad V_2(y) = \sum_{k=1}^{\infty} \frac{v_k}{k} y^k. \tag{1.5}$$

In this case the formal relations

$$\frac{\delta V_1(x)}{\delta V_1(\tilde{x})} = \frac{1}{\tilde{x} - x}, \qquad \frac{\delta V_1'(x)}{\delta V_1(\tilde{x})} = \frac{1}{(\tilde{x} - x)^2}, \tag{1.6}$$

take place, which are used in the derivation of the loop equation. However, here we consider the polynomial case and understand all relations involving the operators $\delta/\delta V_1(x)$ and $\delta/\delta V_2(y)$ in the sense of (1.4).

Consider the resolvents (also understood as formal power series)

$$\mathcal{W}(x) = \frac{1}{N}\left\langle \operatorname{tr} \frac{1}{x - M_1} \right\rangle \quad \text{and} \quad \tilde{\mathcal{W}}(y) = \frac{1}{N}\left\langle \operatorname{tr} \frac{1}{y - M_2} \right\rangle. \tag{1.7}$$

As a corollary of (1.6), the free energy of the two-matrix model (1.1) satisfies the following equations with respect to the coefficients of the polynomial V_1:

$$\frac{\delta F}{\delta V_1(x)} = \mathcal{W}(x), \qquad \frac{\delta F}{\delta V_2(y)} = \tilde{\mathcal{W}}(y), \tag{1.8}$$

which are also valid in the sense of Eqs. (1.4).

Assuming existence of the $1/N^2$ expansion, the highest order contribution F^0 to the free energy was found using Eqs. (1.8) in [16]; it was computed in terms of holomorphic objects associated to the “spectral curve” which arises in $N \to \infty$ limit. The next coefficient F^1 was found in [14] for the case when the spectral curve has genus zero, and in [15] for the case when the genus equals one.

The main result of this paper is an expression for F^1 for an arbitrary genus of “spectral curve”. We compute F^1 in terms of algebro-geometric objects associated to the spectral curve using the loop equations.

The spectral curve is defined by the following equation:

$$\mathcal{E}^0(x, y) := \big(V_1'(x) - y\big)\big(V_2'(y) - x\big) - \mathcal{P}^0(x, y) + 1 = 0, \tag{1.9}$$

where the polynomial of two variables $\mathcal{P}^0(x, y)$ is the zeroth order term in the $1/N^2$ expansion of the polynomial

$$\mathcal{P}(x, y) := \frac{1}{N}\left\langle \operatorname{tr} \frac{V_1'(x) - V_1'(M_1)}{x - M_1} \frac{V_2'(y) - V_2'(M_2)}{y - M_2} \right\rangle; \tag{1.10}$$

the point P of this curve is a pair of complex numbers (x, y) satisfying (1.9).

The spectral curve (1.9) arises together with two meromorphic functions $f(P) = x$ and $g(P) = y$, which project it down to x and y-planes, respectively. These functions have poles only at two points of $\mathcal{L}$, called ∞_f and ∞_g: at ∞_f the function $f(P)$ has a simple pole, and the function $g(P)$ has a pole of order d_1 with singular part equal to $V_1'(f(P))$. At the point ∞_g the function $g(P)$ has a simple pole, and the function

$f(P)$ has a pole of order d_2 with singular part equal to $V_2'(g(P))$. Therefore, we naturally obtain the moduli space $\mathcal{M}$ of triples $(\mathcal{L}, f, g)$, where functions f and g have the pole structure described above. The natural coordinates on this moduli space can be chosen to be coefficients of polynomials V_1 and V_2, and additional g numbers, called "filling fractions" $\epsilon_\alpha = \frac{1}{2\pi i}\oint_{a_\alpha} g\,df$, where a_α are (chosen in some way) canonical a-cycles on $\mathcal{L}$.

Denote the zeros of the differential df by $P_1, \ldots, P_{m_1}$ ($m_1 = d_2 + 2g + 1$) (these points play the role of the ramification points if we realize the spectral curve $\mathcal{L}$ as a branched covering of complex x-plane); the projections of the ramification points on the x-plane are called the branch points, which we denote by $\lambda_j := f(P_j)$. The zeros of the differential dg (the ramification points corresponding to representation of the spectral curve $\mathcal{L}$ as a covering of the complex plane defined by the function $g(P)$) we denote by $Q_1, \ldots, Q_{m_2}$ ($m_2 = d_1 + 2g + 1$); their projections on the y-plane (the branch points) we denote by $\mu_j := g(Q_j)$. We shall assume that the pair of potentials V_1 and V_2 is generic, i.e., all the zeros of the differentials df and dg are simple and distinct.

If is well known [16] how to express all standard algebro-geometrical objects on $\mathcal{L}$ in terms of the previous data. In particular, the canonical meromorphic bidifferential $B(P,Q) = d_P d_Q \ln E(P,Q)$ ($E(P,Q)$ is the prime-form) can be represented as follows:

$$B(P,Q) = \frac{\delta g(P)}{\delta V_1(f(Q))}\bigg|_{f(Q)} df(P)\,df(Q) \tag{1.11}$$

(see [16] for the proof); the bidifferential $B(P,Q)$ is symmetric and has a quadratic pole on the diagonal $P \to Q$ with the following local behavior:

$$B(P,Q) = \left\{\frac{1}{(z(P)-z(Q))^2} + \frac{1}{6}S_B(P) + o(1)\right\} dz(P)\,dz(Q), \tag{1.12}$$

where $z(P)$ is some local coordinate; $S_B(P)$ is the Bergmann projective connection ($S_B(P)$ transforms as a quadratic differential under Möbius transformations of the local coordinate; an appropriate Schwarzian derivative term is added to the projective connection if one makes an arbitrary transformation of the local coordinate).

Consider also the four-differential $D(P,Q) = d_P d_Q^3 \ln E(P,Q)$, which has on the diagonal a pole of the 4th degree: $D(P,Q) = \{6(z(P)-z(Q))^{-4} + O(1)\}\,dz(P)\,(dz(Q))^3$. From $B(P,Q)$ and $D(P,Q)$ it is easy to construct meromorphic normalized (all a-periods vanish) 1-forms on $\mathcal{L}$ with single pole; in particular, if the pole coincides with ramification point P_k, the natural local parameter near P_k is given by $x_k(P) = \sqrt{f(P) - \lambda_k}$. Then the following objects:

$$B(P,P_k) := \frac{B(P,Q)}{dx_k(Q)}\bigg|_{Q=P_k}, \qquad D(P,P_k) := \frac{D(P,Q)}{(dx_k(Q))^3}\bigg|_{Q=P_k} \tag{1.13}$$

are meromorphic normalized 1-forms on $\mathcal{L}$ with a single pole at the point P_k and the following singular parts:

$$\begin{aligned} B(P,P_k) &= \left\{\frac{1}{x_k(P)^2} + \frac{1}{6}S_B(P_k) + o(1)\right\} dx_k(P), \\ D(P,P_k) &= \left\{\frac{6}{x_k(P)^4} + O(1)\right\} dx_k(P) \end{aligned} \tag{1.14}$$

as $P \to P_k$, where $S_B(P_k)$ is the Bergmann projective connection computed at the branch point P_k with respect to the local parameter $x_k(P)$.

In the order $1/N^2$ Eqs. (1.8) look as follows (we write down only the equations with respect to the coefficients of the polynomial V_1):

$$\frac{\delta F^1}{\delta V_1(f(P))} = -Y^1(P), \tag{1.15}$$

where Y^1 is the $1/N^2$ contribution to the resolvent $\mathcal{W}$. The function Y^1 can be computed using the loop equations [14], which leads to the following expression:

$$\begin{aligned} &Y^{(1)}(P)\,df(P) \\ &\quad = \sum_{k=1}^{m_1} \left\{ -\frac{1}{96 g'(P_k)} D(P, P_k) + \left[\frac{g'''(P_k)}{96 g'^2(P_k)} - \frac{S_B(P_k)}{24 g'(P_k)} \right] B(P, P_k) \right\}. \end{aligned} \tag{1.16}$$

The solution of Eqs. (1.15), (1.16) which is invariant with respect to the projection change (i.e., which satisfies also the required equations with respect to the coefficients of the polynomial V_2), looks as follows:

$$F^1 = \frac{1}{24} \ln \left\{ \tau_f^{12} (v_{d_2+1})^{1-\frac{1}{d_2}} \prod_{k=1}^{m_1} dg(P_k) \right\} + \frac{d_2+3}{24} \ln d_2, \tag{1.17}$$

where τ_f is the so-called Bergmann tau-function on the Hurwitz space. The Bergmann tau-function is defined as the (unique up to an additive constant) solution of the following system of equations with respect to the branch points λ_k:

$$\frac{\partial}{\partial \lambda_k} \ln \tau_f = -\frac{1}{12} S_B(P_k). \tag{1.18}$$

The Bergmann tau-function (1.18) appears in many important problems: it coincides with the isomonodromic tau-function of Hurwitz Frobenius manifolds [18], and gives the main contribution to the G-function (the solution of Getzler equation) of these Frobenius manifolds; it gives the most non-trivial term in the Jimbo–Miwa tau-function corresponding to a Riemann–Hilbert problem with regular singularities and quasi-permutation monodromies. Finally, its modulus square essentially coincides with the determinant of Laplace operator in metrics with conic singularities over Riemann surfaces [19]. The solution of the system (1.18) was found in [20] and can be described as follows.

Introduce the divisor $(df) = -2\infty_f - (d_2+1)\infty_g + \sum_{k=1}^{m_1} P_k := \sum_{k=1}^{m_1+2} r_k D_k$. Choose some initial point $P \in \hat{\mathcal{L}}$ and consider the vector of Riemann constants K^P and the Abel map $\mathcal{A}_\alpha(Q) = \int_P^Q w_\alpha$ (w_α are normalized holomorphic 1-forms on $\mathcal{L}$). Since all the zeros of the differential df have multiplicity 1, we can always choose the fundamental cell $\hat{\mathcal{L}}$ of the universal covering of the spectral curve $\mathcal{L}$ in such a way that $\mathcal{A}((df)) = -2K^P$ (for an arbitrary choice of the fundamental domain these two vectors coincide only up to an integer combination of the periods of the holomorphic differentials); the Abel map is computed along a path which does not intersect the boundary of $\hat{\mathcal{L}}$.

The main ingredient of the Bergmann tau-function is the following holomorphic multivalued $g(1-g)/2$-differential $\mathcal{C}(P)$ on $\mathcal{L}$:

$$\mathcal{C}(P) := \frac{1}{W(P)} \sum_{\alpha_1,\ldots,\alpha_g=1}^{g} \frac{\partial^g \Theta(K^P)}{\partial z_{\alpha_1}\ldots\partial z_{\alpha_g}} w_{\alpha_1}(P)\ldots w_{\alpha_g}(P), \tag{1.19}$$

where

$$W(P) := \det_{1\leqslant\alpha,\beta\leqslant g} \left\| w_\beta^{(\alpha-1)}(P) \right\| \tag{1.20}$$

denotes the Wronskian determinant of the holomorphic differentials. Introduce also the quantity $\mathcal{Q}$ defined by the expression

$$\mathcal{Q} = \left[df(P)\right]^{\frac{g-1}{2}} \mathcal{C}(P) \prod_{k=1}^{m+2} \left[E(P,D_k)\right]^{\frac{(1-g)r_k}{2}}; \tag{1.21}$$

this combination is independent of the point $P \in \mathcal{L}$. Then the Bergmann tau-function (1.18) on the Hurwitz space is given by the following expression:

$$\tau_f = \mathcal{Q}^{2/3} \prod_{k,l=1,\,k<l}^{m+n} \left[E(D_k,D_l)\right]^{\frac{r_k r_l}{6}}; \tag{1.22}$$

together with (1.17) this gives a formula for the genus one correction in Hermitian two-matrix model.

If the potential V_2 is quadratic, the integration with respect to M_2 in (1.1) can be taken explicitly, and the free energy (1.17) gives rise to the free energy of one-matrix model. The spectral curve $\mathcal{L}$ in this case becomes hyperelliptic, and the formula (1.17) turns into (after using the expression for τ_f obtained in [23])

$$F^1 = \frac{1}{24} \ln\left\{ \Delta^3 (\det \mathbf{A})^{12} \prod_{k=1}^{2g+2} g'(\lambda_k) \right\}, \tag{1.23}$$

where λ_k, $k = 1,\ldots,2g+2$, are the branch points of $\mathcal{L}$; Δ is their Wronskian determinant; $\mathbf{A}$ is the matrix of a-periods of the non-normalized holomorphic differentials on $\mathcal{L}$.

The paper is organized as follows. In Section 2, following [14], we write down the loop equations for the two-matrix model, and discuss the spectral curve and associated objects which arise in the zeroth order in $1/N^2$ expansion. Here we derive also new variational formulas, which will be used later in computation of $1/N^2$ correction to free energy. In Section 3 we solve the loop equations in $1/N^2$ approximation. Here we also express F^1 in terms of the Bergmann tau-function on Hurwitz spaces introduced in [18,26]. In Section 4 we recall an explicit expression for the Bergmann tau-function [20], and find its transformation law under the change of projection of the spectral curve. This allows to get a formula for F^1 which satisfies the full set of variational equations with respect to the coefficients of the polynomials V_1 and V_2. In Section 5 we derive variational equations of F^1 with respect to filling fractions. In Section 6 we discuss the links between F^1 and other related objects: the determinant of Laplace operator, the G-function of Frobenius manifolds and the isomonodromic tau-function of Fuchsian system with quasi-permutation monodromies. Finally, in Section 7 we consider the simplest partial cases, when the spectral curve is either rational ("one-cut" case) or elliptic ("two-cut" case); here we also describe reduction to the one-matrix model.

2. Loop equations: the leading term

Introduce the function

$$Y(x) = V_1'(x) - \mathcal{W}(x). \tag{2.1}$$

In terms of the function Y Eqs. (1.8) for the free energy can be written as follows:

$$\frac{\delta F}{\delta V_1(x)} = V_1'(x) - Y(x); \tag{2.2}$$

as well as (1.8), these equations are valid in the sense of (1.4).

To make use of the variational formula (2.2) we need to get some information about the function $Y(x)$. This information is in principle contained in the loop equations, which follow from the reparametrization invariance of the partition function (1.1) (see [14] for details). To write down the loop equations, in addition to the resolvent $\mathcal{W}(x)$ (1.7), we need to introduce the following objects:

- the polynomial $\mathcal{P}(x, y)$:

$$\mathcal{P}(x, y) := \frac{1}{N}\left\langle \operatorname{tr} \frac{V_1(x) - V_1(M_1)}{x - M_1} \frac{V_2(y) - V_2(M_2)}{y - M_2} \right\rangle; \tag{2.3}$$

- the polynomial $\mathcal{E}(x, y)$

$$\mathcal{E}(x, y) := \big(V_1(x) - y\big)\big(V_2(y) - x\big) - \mathcal{P}(x, y) + 1; \tag{2.4}$$

- the function $\mathcal{U}(x, y)$, which is a polynomial in y:

$$\mathcal{U}(x, y) := \frac{1}{N}\left\langle \operatorname{tr} \frac{1}{x - M_1} \frac{V_2'(y) - V_2'(M_2)}{y - M_2} \right\rangle; \tag{2.5}$$

- the function $\mathcal{U}(x, y, z)$, which is also a polynomial in y:

$$\begin{aligned} \mathcal{U}(x, y, z) &:= \frac{\delta \mathcal{U}(x, y)}{\delta V_1(z)} \\ &= \left\langle \operatorname{tr} \frac{1}{x - M_1} \frac{V_2'(y) - V_2'(M_2)}{y - M_2} \operatorname{tr} \frac{1}{z - M_1} \right\rangle - N^2 \mathcal{U}(x, y)\mathcal{W}(z). \end{aligned} \tag{2.6}$$

Now we can write down the loop equation

$$\mathcal{U}(x, y) = x - V_2'(y) + \frac{\mathcal{E}(x, y)}{y - Y(x)} - \frac{1}{N^2} \frac{\mathcal{U}(x, y, x)}{y - Y(x)}, \tag{2.7}$$

which arises as a corollary of the reparametrization invariance of the matrix integral (1.1) [14].

The residue of (2.7) at $y = Y(x)$ leads to the following loop equation (for polynomials of degree 3 this equation was first derived in [5]) for the function $Y(x) := V_1'(x) - \mathcal{W}(x)$:

$$\mathcal{E}^0\big(x, Y(x)\big) = \frac{1}{N^2}\mathcal{U}\big(x, Y(x), x\big). \tag{2.8}$$

To use the loop equations effectively we need to consider the $1/N^2$ expansion of all their ingredients.

2.1. Leading order term: algebro-geometric framework

Assume that the function Y admits an expansion into a power series in $1/N^2$:

$$Y(x) = Y^0 + \frac{1}{N^2}Y^1 + \cdots. \tag{2.9}$$

Then the master loop equation (2.8) in the leading order turns into an algebraic equation in two variables, x and $Y^{(0)}(x)$:

$$\mathcal{E}\big(x, Y^0(x)\big) = 0,$$

where

$$\mathcal{E}^0(x, y) = \big(V_1'(x) - y\big)\big(V_2'(y) - x\big) - \mathcal{P}^0(x, y) + 1. \tag{2.10}$$

The polynomial equation

$$\mathcal{E}^0(x, y) = 0 \tag{2.11}$$

defines an algebraic curve $\mathcal{L}$, which we call "spectral curve"; denote its genus by g (if the spectral curve is non-singular, it has "maximal genus" equal to $d_1 d_2 - 1$); a point P of this curve is a pair of complex numbers (x, y) satisfying the polynomial equation (2.11). Therefore, Y^0 can be considered as a multi-valued function of x. The curve $\mathcal{L}$ is naturally equipped with two meromorphic functions: the function $f(P) = x$ and the function $g(P) = y$ ($\equiv Y^0(x)$). Since the polynomial $\mathcal{P}$ (2.3) and the function $\mathcal{E}$ (2.4) are symmetric with respect to the substitution $x \leftrightarrow y$, $V_1 \leftrightarrow V_2$, the same algebraic curve appears if we write down the loop equations for $X(y) := V_2'(y) - \frac{\delta F}{\delta V_2(y)}$.

Analytical properties of the functions $f(P)$ and $g(P)$ on $\mathcal{L}$ are well known (see [16, 17] and references therein). Namely, $f(P)$ and $g(P)$ are meromorphic functions on $\mathcal{L}$ having poles only at the marked points ∞_f and ∞_g with the following pole structure: the function $f(P)$ has a simple pole at ∞_f and a pole of order d_1 at ∞_g; the function $g(P)$ has a simple pole at ∞_g and a pole of order d_2 at ∞_f. Therefore, near ∞_f we can write down the singular part of $g(P)$ as a polynomial of $f(P)$; near ∞_g we can represent the singular part of $f(P)$ as a polynomial of $g(P)$; the coefficients of these polynomials coincide with the coefficients of the polynomials V_1' and V_2', respectively:

$$g(P) = V_1'\big(f(P)\big) - \frac{1}{f(P)} + O\big(f^{-2}(P)\big) \quad \text{as } P \to \infty_f, \tag{2.12}$$

$$f(P) = V_2'\big(g(P)\big) - \frac{1}{g(P)} + O\big(g^{-2}(P)\big) \quad \text{as } P \to \infty_g. \tag{2.13}$$

The dimension of the moduli space of triples $(\mathcal{L}, f, g)$ satisfying these conditions equals $d_1 + d_2 + g + 2$.

Let us choose on $\mathcal{L}$ a canonical basis of cycles (a_α, b_α). Then coordinates on the space $\mathcal{M}$ can be chosen as follows:

- $d_1 + 1$ coefficients $u_1, \ldots, u_{d_1+1}$ of the polynomial V_1';
- $d_2 + 1$ coefficients $v_1, \ldots, v_{d_2+1}$ of the polynomial V_2';
- "filling fractions"

$$\epsilon_\alpha := \frac{1}{2\pi i} \oint_{a_\alpha} g\, df. \tag{2.14}$$

In strictly physical situation the potentials V_1 and V_2 should be such that, considering $\mathcal{L}$ as a branched covering defined by the function f, one can single out the "physical" sheet (which includes the point ∞_f) such that all a-cycles lie on this sheet and each a-cycle encircles exactly one branch cut (all corresponding branch points must be real if the potentials V_1 and V_2 are real). Similar requirement comes from the g-projection of $\mathcal{L}$. However, here we do not impose these "physical" requirements, i.e., we consider the "analytical continuation" of the physical sector, in the spirit of [27].

Nevertheless, the sheet of the curve $\mathcal{L}$ (realized as (d_2+1)-sheeted branched covering by function f), which contains the point ∞_f, is called the "physical" sheet; the physical sheet is well-defined at least in some neighborhood of ∞_f. Fixing some splitting of $\mathcal{L}$ into $d_2 + 1$ sheets, we denote by $x^{(k)}$ $(k = 1, \ldots, d_2 + 1)$ the point of $\mathcal{L}$ belonging to the kth sheet such that $f(x^{(k)})x = x$; we assume that the point $x^{(1)}$ belongs to the physical sheet of $\mathcal{L}$, i.e., $x^{(1)} \to \infty_f$ as $x \to \infty$.

The polynomial $\mathcal{E}^0(x, y)$ defining the spectral curve $\mathcal{L}$ (2.11) can also be rewritten as follows:

$$\mathcal{E}^0(x, y) = -v_{d_2+1} \prod_{k=1}^{d_2+1} \bigl(y - g\bigl(x^{(k)}\bigr)\bigr). \tag{2.15}$$

The proof of (2.15) is simple: the function $\mathcal{E}^0$ is given by (2.10); since $\mathcal{P}^0$ is a polynomial of degree $d_2 - 1$ with respect to y, the function $\mathcal{E}^0$ is a polynomial of degree $d_2 + 1$ in y; its zeros are given by $Y^0(x^{(k)})$ according to the definition of the points $x^{(k)}$. Comparison of the coefficients in front of y^{d_2+1} leads to (2.15).

2.2. Some variational formulas

If a Riemann surface is realized as a branched covering of the Riemann sphere, the branch points can be used as natural local coordinates on the moduli space. Dependence of normalized holomorphic differentials, the matrix of b-periods and the canonical meromorphic bidifferential on the branch points is given by Rauch variational formulas [7] (for a simple proof see [21]). However, on our moduli space the set of natural coordinates is given by the coefficients of polynomials V_1 and V_2 and the filling fractions. To differentiate all interesting objects with respect to these coordinates we need to know the matrix of derivatives of the branch points with respect to coefficients of V_1, V_2 and filling fractions. This matrix was computed in [16]; below we re-derive some of these formulas, and prove new variational formulas, required in our context.

In [16] Eqs. (2.2), together with analogous equations with respect to $V_2(y)$, were solved in the leading term, i.e., it was found a solution of the system

$$\left.\frac{\delta F^0}{\delta V_1(f(P))}\right|_{f(P)} = V_1'\big(f(P)\big) - g(P),$$
$$\left.\frac{\delta F^0}{\delta V_2(g(P))}\right|_{g(P)} = V_2'\big(g(P)\big) - f(P)$$

which a posteriori turns out to satisfy also the following equations with respect to the filling fractions:

$$\frac{\partial F^0}{\partial \epsilon_\alpha} = \Gamma_\alpha := \oint_{b_\alpha} g(P)\,df(P).$$

To find a solution of Eqs. (2.2) in order $1/N^2$ (which would also satisfy a similar set of equations with respect to $V_2(y)$) we shall need the following.

Lemma 1. *The following variational formulas take place*:

$$-\frac{\delta\lambda_k}{\delta V_1(f(P))} g'(P_k)\,df(P) = B(P, P_k), \tag{2.16}$$
$$\left.\frac{\delta\{g'(P_k)\}}{\delta V_1(f(P))}\right|_{f(P)} df(P) = \frac{1}{4}\left\{D(P, P_k) - \frac{g'''(P_k)}{g'(P_k)} B(P, P_k)\right\}, \tag{2.17}$$

where prime denotes derivative with respect to the local parameter $x_k := x_k(Q) = \sqrt{f(Q) - \lambda_k}$.

Proof. We start from the formula (1.11):

$$B(P, Q) = \left.\frac{\delta g(P)}{\delta V_1(f(Q))}\right|_{f(Q)} df(P)\,df(Q). \tag{2.18}$$

Let us rewrite this formula in the limit $Q \to P_k$ using the local parameter $x_k(Q)$. First we notice that for any coordinate t on our moduli space the following identity takes place:

$$g_t(Q)|_{f}(Q)\,df(Q) = g_t(Q)|_{x_k(Q)}\,df(Q) - f_t(Q)|_{x_k(Q)}\,dg(Q), \tag{2.19}$$

which follows from differentiation of the function $g(t, f(x_k, t))$ with respect to variable t using the chain rule. In particular,

$$\left.\frac{\delta g(Q)}{\delta V_1(f(P))}\right|_{f}(Q)\,df(Q) = \left.\frac{\delta g(Q)}{\delta V_1(f(P))}\right|_{x_k(Q)} df(Q) - \left.\frac{\delta g(Q)}{\delta V_1(f(P))}\right|_{x_k(Q)} dg(Q). \tag{2.20}$$

Consider now the first several terms of the local expansion of $g(Q)$, $dg(Q)$ and $B(P,Q)$ as $Q \to P_k$:

$$g(Q) = g(P_k) + g'(P_k)x_k + \cdots, \tag{2.21}$$

$$dg(Q) = \left\{ g'(P_k) + g''(P_k)x_k + \frac{1}{2}g'''(P_k)x_k^2 + \cdots \right\} dx_k, \tag{2.22}$$

$$B(P,Q) = \left\{ B(P,P_k) + B'(P,P_k)x_k + \frac{1}{2}B''(P,P_k)x_k^2 + \cdots \right\} dx_k. \tag{2.23}$$

Taking into account that $f(Q) = x_k^2 + \lambda_k$, and substituting these relations into (2.20), we get in the zeroth order the formula (2.16).

Coincidence of coefficients in front of x_k in (2.20) gives rise to the following relation which defines the dependence of $g(P_k)$ on $\{u_k\}$:

$$\left\{ 2\frac{\delta g(P_k)}{\delta V_1(f(P))} - \frac{\delta \lambda_k}{\delta V_1(f(P))} g''(P_k) \right\} df(P) = B'(P,P_k); \tag{2.24}$$

we present this relation only for completeness, since it will not be used below.

Finally, collecting the coefficients in front of x_k^2, we get

$$2\frac{\delta g'(P_k)}{\delta V_1(f(P))} - \frac{1}{2}\frac{\delta \lambda_k}{\delta V_1(f(P))} g'''(P_k) = \frac{1}{2}\frac{B''(P,P_k)}{df(P)},$$

which leads to (2.17) after using (2.16). □

3. Solution of loop equation in $1/N^2$ approximation

The main goal of this paper is to find a solution of the following equation:

$$\frac{\delta F^1}{\delta V_1(x)} = -Y^1(x), \tag{3.1}$$

where $Y^1(x)$ is determined from the $1/N^2$ expansion of the loop Eq. (2.8). Eq. (3.1) is valid in a neighborhood of the point ∞_f, i.e., in a neighborhood of the point $x = \infty$ on the "physical" (with respect to the variable x) sheet of the spectral curve $\mathcal{L}$. The function F^1 should also satisfy the equation

$$\frac{\delta F^1}{\delta V_2(y)} = -X^1(y), \tag{3.2}$$

where the function $X^1(y)$ should be found from the loop equation written down with respect to the matrix M_2 in a neighborhood of the point ∞_g. We shall first solve Eqs. (3.1), and then check the symmetry of the obtained expression with respect to the change of projection $f \leftrightarrow g$.

To express Y^1 in terms of the objects associated to the spectral curve $\mathcal{L}$ we consider the $1/N^2$ term of the master loop equation (2.8). We have

$$\mathcal{E}\big(x, Y(x)\big) = \mathcal{E}^0\left(f(P), g(P) + \frac{1}{N^2}Y^1(P) + \cdots \right)$$

$$+\frac{1}{N^2}\mathcal{E}^1\big(f(P),g(P)\big)+\cdots \tag{3.3}$$

as $P\to\infty_f$, where, as before, in a neighborhood of ∞_f, $f(P)=x$; $g(P)=Y^0(x)$. The $1/N^2$ expansion of $\mathcal{E}(x,y)$ looks as follows:

$$\mathcal{E}(x,y)=\mathcal{E}^0(x,y)+\frac{1}{N^2}\mathcal{E}^1(x,y)+\cdots; \tag{3.4}$$

since $\mathcal{E}^1(x,y)=-\mathcal{P}^1(x,y)$, we can further rewrite this expression in a neighborhood of the point ∞_f as follows:

$$\begin{aligned}\mathcal{E}\big(x,Y(x)\big)={}&\mathcal{E}^0\big(f(P),g(P)\big)\\&+\frac{1}{N^2}\big\{\mathcal{E}^1\big(f(P),g(P)\big)+Y^1(P)\mathcal{E}^0_y\big(f(P),g(P)\big)\big\}+\cdots.\end{aligned} \tag{3.5}$$

Therefore, the $1/N^2$ term of the master loop equation (2.8) gives

$$\mathcal{U}^0\big(f(P),g(P),f(P)\big)=\mathcal{E}^1\big(f(P),g(P)\big)+Y^1(P)\mathcal{E}^0_y\big(f(P),g(P)\big)$$

as $P\to\infty_f$, or

$$Y^1(P)=\frac{\mathcal{U}^0(f(P),g(P),f(P))+\mathcal{P}^1(f(P),g(P))}{\mathcal{E}^0_y(f(P),g(P))}. \tag{3.6}$$

To make this formula more explicit we need to express $\mathcal{U}^0(f(P),g(P),f(P))$ in terms of known objects using the loop equation (2.7). According to the definition of $\mathcal{U}^0(x,y,z)$ we have:

$$\mathcal{U}^0(x,y,z)=-\frac{\delta\mathcal{U}^0(x,y)}{\delta V_1(z)}. \tag{3.7}$$

On the other hand, the zeroth order term of (2.7) gives

$$\mathcal{U}^0(x,y)=x-V_2'(y)+\frac{\mathcal{E}^0(x,y)}{y-g(x^{(1)})} \tag{3.8}$$

(as before, $x^{(1)}$ denotes a point on the physical sheet of $\mathcal{L}$). Therefore,

$$\mathcal{U}^0(x,y,z)=-\frac{\delta\mathcal{E}^0(x,y)/\delta V_1(z)}{y-g(x^{(1)})}-\frac{\mathcal{E}^0(x,y)}{(y-g(x^{(1)}))^2}\frac{\delta g(x^{(1)})}{\delta V_1(z)}. \tag{3.9}$$

Using the form (2.15) of the polynomial $\mathcal{E}^0(x,y)$, we can further rewrite this expression as follows:

$$\frac{\delta\mathcal{E}^0(x,y)}{\delta V_1(z)}=-\mathcal{E}^0(x,y)\sum_{k=1}^{d_2+1}\frac{\delta g(x^{(k)})}{\delta V_1(z)}\frac{1}{y-g(x^{(k)})}. \tag{3.10}$$

Substituting this formula into (2.15), we get

$$\mathcal{U}^0(x,y,z)=\frac{\mathcal{E}^0(x,y)}{y-g(x^{(1)})}\sum_{k=2}^{d_2+1}\frac{\delta g(x^{(k)})}{\delta V_1(z)}\frac{1}{y-g(x^{(k)})}. \tag{3.11}$$

Choosing $z = x = f(P)$ and taking the limit $y \to g(x^{(1)})$, we have

$$\mathcal{U}^0\big(f(P), g(P), f(P)\big) = \mathcal{E}_y^0\big(f(P), g(P)\big) \sum_{k=2}^{d_2+1} \frac{\delta g(x^{(k)})}{\delta V_1(f(P))} \frac{1}{g(P) - g(x^{(k)})} \tag{3.12}$$

as $P \equiv x^{(1)} \to \infty_f$. Now (3.6) can be rewritten as follows:

$$Y^1(P) = \frac{\mathcal{P}^1(f(P), g(P))}{\mathcal{E}_y^0(f(P), g(P))} + \sum_{Q \neq P:\ f(Q)=f(P)} \frac{\delta g(Q)}{\delta V_1(f(P))} \frac{1}{g(P) - g(Q)} \tag{3.13}$$

as $P \to \infty_f$; this expression can be further transformed, using the formula (1.11) for the bidifferential $B(P, Q)$:

$$\begin{aligned} Y^1(P)\, df(P) &= \frac{\mathcal{P}^1(f(P), g(P))}{\mathcal{E}_y^0(f(P), g(P))}\, df(P) \\ &\quad + \sum_{Q \neq P:\ f(Q)=f(P)} \frac{B(P, Q)}{df(Q)} \frac{1}{g(P) - g(Q)}; \end{aligned} \tag{3.14}$$

now we see that the 1-form $Y^1(P)\, df(P)$ can be analytically continued from a neighborhood of ∞_f to the whole $\mathcal{L}$.

Lemma 2. *Let the spectral curve $\mathcal{L}$ (2.11) be non-singular. Then the 1-form $Y^1(P)\, df(P)$ (3.14) is a meromorphic 1-form on the spectral curve $\mathcal{L}$ which has poles (up to fourth order) only at the branch points P_k, i.e., at the zeros of differential $df(P)$.*

Proof. Let us verify the non-singularity of the first term,

$$\frac{\mathcal{P}^1(f(P), g(P))}{\mathcal{E}_y^0(f(P), g(P))}\, df(P) \tag{3.15}$$

of the expression (3.14), everywhere on $\mathcal{L}$. For finite $f(P)$ the 1-form (3.15) can be singular only at the zeros of $\mathcal{E}_y^0(f(P), g(P))$, which, if the curve $\mathcal{L}$ is non-singular, are by definition the branch points P_k; these zeros are of the first order and are canceled by the zeros of $df(P)$ at the branch points.

To study the behavior of (3.15) at ∞_f and ∞_g we mention that the polynomial $\mathcal{P}(x, y)$ (2.3) (and, therefore, also its first correction $\mathcal{P}^1(x, y)$) is of degree $d_1 - 1$ with respect to x and $d_2 - 1$ with respect to y. However, we can say a bit more about $\mathcal{P}^1(x, y)$. Namely, the coefficient of $\mathcal{P}(x, y)$ in front of $x^{d_1-1} y^{d_2-1}$ equals $u_{d_1+1} v_{d_2+1}$, which does not have any higher corrections. Therefore, the coefficient of the polynomial $\mathcal{P}^1(x, y)$ in front of $x^{d_1-1} y^{d_2-1}$ vanishes.

Now consider the behavior of the 1-form (3.15) near ∞_f. We have

$$\begin{aligned} \mathcal{E}_y^0\big(f(P), g(P)\big) &= -\big(V_2'\big(g(P)\big) - f(P)\big) - \big(V_1'\big(f(P)\big) - g(P)\big) V_2''\big(g(P)\big) \\ &\quad - \mathcal{P}_y^0\big(f(P), g(P)\big); \end{aligned}$$

this expression has a pole of order $d_1 d_2$ near ∞_f as a corollary of the asymptotics (2.12) of the function $g(P)$ near ∞_f. The 1-form $df(P)$ has a pole of second order at

∞_f. The most singular contribution of $\mathcal{P}^1(f(P), g(P))$ near ∞_f comes from the term $f^{d_1-2}(P)g^{d_2-1}(P)$; it has pole of order $d_1 - 2 + d_1(d_2 - 1) = d_1d_2 - 2$. Summing up all degrees, we see that (3.15) is non-singular near ∞_f.

Consider the 1-form (3.15) near ∞_g. At ∞_g the differential $df(P)$ has a pole of order $d_2 + 1$; the main contribution to $\mathcal{E}_y^0(f(P), g(P))$ is given by the term $(V_1'(f(P)) - g(P))V_2''(g(P))$, which has a pole of order $d_1d_2 + d_2 - 1$. Finally, the main contribution to $\mathcal{P}^1(f(P), g(P))$ comes from the term $g^{d_2-2}(P)f^{d_1-1}(P)$, which has a pole of order $d_1d_2 - 2$. Summing up all degrees, we see that (3.15) is non-singular at ∞_g.

Consider now the second term of (3.14)

$$\sum_{Q\neq P:\ f(Q)=f(P)} \frac{B(P,Q)}{df(Q)} \frac{1}{g(P)-g(Q)}. \tag{3.16}$$

The bidifferential $B(P, Q)$ is singular (has second order poles) only at coinciding arguments, which now means that P coincides with Q and with one of the branch points P_k. The denominator $g(P) - g(Q)$ also vanishes only if P coincides with Q, (i.e., again both of them coincide with one of the branch points P_k). It is slightly more complicated to see that zeros of $df(Q)$ do not produce any poles outside of P_k. Obviously, $df(Q)$ is singular if $P \to P_k$ and $Q = P^*$, where P^* is another point such that $f(P^*) = f(P)$ and $P^* \to P_k$ as $P \to P_k$. However, $df(Q)$ is also singular if Q coincides with one of the branch points P_k, while P remains on some other sheet, and does not tend to P_k as $Q \to P_k$. In this case in the sum (3.16) we have two singular terms (with poles of first order), which correspond to Q and Q^*; however, the residues of these terms just differ by sign, and, therefore, the total sum (3.16) remains finite outside the branch points P_k and infinities ∞_f and ∞_g.

As $P \to \infty_f$, all corresponding points Q in (3.16) tend to ∞_g; at all of these points the differential $df(Q)$ has poles of order $d_2 + 2$; all other terms remain non-singular and non-vanishing. Therefore, (3.16) has a zero of order $d_2 + 1$ at ∞_f.

As $P \to \infty_g$, the situation is slightly more complicated. Let us enumerate the sheets of $\mathcal{L}$ such, that $x^{(1)} \to \infty_f$ and $x^{(2)}, \ldots, x^{(d_2+1)} \to \infty_g$, as $x \to \infty$. Let us also choose $P := x^{(d_2+1)}$. Then (3.16) can be split as follows:

$$\frac{B(x^{(1)}, x^{(d_2+1)})}{df(x^{(1)})} \frac{1}{g(x^{(d_2+1)}) - g(x^{(1)})} + \sum_{j=2}^{d_2} \frac{B(x^{(j)}, x^{(d_2+1)})}{df(x^{(j)})} \frac{1}{g(x^{(d_2+1)}) - g(x^{(j)})}. \tag{3.17}$$

As $x \to \infty$, the first term in (3.17) has a zero of order two ($df(x^{(1)})$ has a pole of order two, other multipliers remain non-singular and non-vanishing). The bidifferential $B(P, Q)$ has a pole of second order as $x \to \infty$ in each term of the sum in (3.17). However, $df(x^{(j)})$ has a pole of order $d_2 + 1$, and $g(x^{(d_2+1)}) - g(x^{(j)})$ has a simple pole as $x \to \infty$. Therefore, the whole expression (3.17) is non-singular (and even vanishing) as $x \to \infty$. □

Remark 1. The condition of non-singularity of the spectral curve (2.11) made in Lemma 2 means in physical language that the spectral curve has maximal possible genus equal to

$d_1 d_2 - 1$ for given degrees of polynomials V_1 and V_2. If the genus of the spectral curve is less than the maximal genus, the spectral curve must be singular; in this case the non-singularity of the 1-form $Y^1(P)\,df(P)$ at the double points cannot be verified rigorously. However, this non-singularity is suggested by physical consideration: since we assume that at the double points the matrix M_1 does not have any eigenvalues in the large N limit (i.e., corresponding filling fractions are equal to zero), there is no physical reason for corresponding resolvents to be singular at these points. Therefore, in the sequel we shall assume that $Y^1(P)\,df(P)$ is non-singular outside of branch points of $\mathcal{L}$ both for maximal and non-maximal genus. We should mention that this assumption was also made (explicitly or implicitly) in the previous papers [8,9,13,15].

The singular parts of $Y^1(P)\,df(P)$ at the branch points P_k can be found from (3.14). If, say, $P \to P_k$, then the only term in (3.14) which contributes to singular part at P_k corresponds to $Q = P^*$. Thus

$$Y^1(P)\,df(P) = \frac{B(P,P^*)}{df(P^*)}\frac{1}{g(P)-g(P^*)} + O(1) \quad \text{as } P \to P_k. \tag{3.18}$$

Consider the local expansion of all ingredients of this expression as $P \to P_k$ in terms of the local parameter $x_k(P) = \sqrt{f(P)-\lambda_k}$:

$$\begin{aligned}
&g(P) = g(P_k) + x_k(P)g'(P_k) + \frac{1}{2}x_k^2(P)g''(P_k) + \frac{1}{6}x_k^3(P)g'''(P_k) + \cdots,\\
&g\big(P^*\big) = g(P_k) - x_k(P)g'(P_k) + \frac{1}{2}x_k^2(P)g''(P_k) - \frac{1}{6}x_k^3(P)g'''(P_k) + \cdots,\\
&df\big(P^*\big) = 2x_k(P)\,dx_k(P),\\
&B\big(P,P^*\big) = \left(\frac{1}{(2x_k(P))^2} + \frac{1}{6}S_B(P_k) + \cdots\right)dx_k(P)\big(-dx_k(P)\big).
\end{aligned}$$

We have

$$\frac{1}{g(P)-g(P^*)} = \frac{1}{2x_k(P)g'(P_k)}\left(1 - \frac{x_k(P)^2}{6}\frac{g'''(P_k)}{g'(P_k)}\right) + \cdots,$$

and, as $P \to P_k$,

$$\begin{aligned}
&\frac{B(P,P^*)}{df(P^*)}\frac{1}{g(P)-g(P^*)}\\
&\quad = \left\{-\frac{1}{16x_k^4(P)g'(P_k)} + \left(\frac{1}{96}\frac{g'''(P_k)}{g'^2(P_k)} - \frac{S_B}{24g'(P_k)}\right)\frac{1}{x_k^2(P)} + O(1)\right\}dx_k(P).
\end{aligned} \tag{3.19}$$

Since, according to our assumption, the 1-form $Y^1(P)\,df(P)$ is non-singular on $\mathcal{L}$ outside of the branch points, we can express this 1-form in terms of differentials $B(P,P_k)$ and $D(P,P_k)$ (1.13) using their behavior near P_k:

$$\begin{aligned}
&Y^{(1)}(P)\,df(P)\\
&\quad = \sum_{k=1}^{m_1}\left\{-\frac{1}{96g'(P_k)}D(P,P_k) + \left[\frac{g'''(P_k)}{96g'^2(P_k)} - \frac{S_B(P_k)}{24g'(P_k)}\right]B(P,P_k)\right\};
\end{aligned} \tag{3.20}$$

as a result we rewrite Eq. (3.1) for F^1 as follows:

$$\frac{\delta F^1}{\delta V_1(f(P))} df(P) \\ = \sum_{k=1}^{m_1} \left\{ \frac{1}{96 g'(P_k)} D(P, P_k) + \left[-\frac{g'''(P_k)}{96 g'^2(P_k)} + \frac{S_B(P_k)}{24 g'(P_k)} \right] B(P, P_k) \right\}. \tag{3.21}$$

Proposition 1. *The general solutions F^1 of the system (3.21) can be written as follows*:

$$F^1 = \frac{1}{2} \ln \tau_f + \frac{1}{24} \ln \left\{ \prod_{k=1}^{m_1} g'(P_k) \right\} + C\big(\{v_k\}, \{\epsilon_\alpha\}\big), \tag{3.22}$$

where $C(\{v_k\}, \{\epsilon_\alpha\})$ is a function on our moduli space depending only on coefficients of the polynomial V_2 and filling fractions $\{\epsilon_\alpha\}$; function τ_f (the Bergmann tau-function on Hurwitz space) is defined by the system of equations with respect to branch points $\{\lambda_k\}$:

$$\frac{\partial}{\partial \lambda_k} \ln \tau_f = -\frac{1}{12} S_B(P_k); \tag{3.23}$$

function τ_f depends on coordinates $\{u_k,\ v_k,\ \epsilon_\alpha\}$ as a composite function.

Proof. The derivative of τ_f with respect to $V_1(f(P))$ is computed via the chain rule using the variational formula (2.16); derivatives of $g'(P_k)$ with respect to $V_1(f(P))$ are given by (2.17). Collecting all these terms together we see that the derivative of (3.22) coincides with (3.21). □

Therefore, to compute F^1 it remains to find the Bergmann tau-function τ_f and to make sure that the "constant" $C(\{v_k\}, \{\epsilon_\alpha\})$ is chosen such that the final expression is symmetric with respect to the change of "projection", i.e., that F^1 satisfies also Eqs. (3.2).

4. F^1 and Bergmann tau-function on Hurwitz spaces

4.1. Bergmann tau-function on Hurwitz spaces

Here, following [20], we discuss the Bergmann tau-function on Hurwitz spaces for the stratum of the Hurwitz space which arises in the application to the two-matrix model.

The Hurwitz space $H_{g,N}$ is the space of equivalence classes of pairs $(\mathcal{L}, f)$, where $\mathcal{L}$ is a compact Riemann surface of genus g and f is a meromorphic functions of degree N. The Hurwitz space is stratified according to multiplicities of poles of function f. By $H_{g,N}(k_1, \cdots, k_n)$, where $k_1 + \cdots + k_n = N$, we denote the stratum of $H_{g,N}$ consisting of meromorphic functions which have n poles on $\mathcal{L}$ with multiplicities $k_1, \ldots, k_n$. (In applications to two-matrix model we need to study the tau-function on the stratum $H_{g,N}(1, N-1)$, when the function f has only two poles: one simple pole and one pole of order $N-1$.)

Suppose that all critical points of the function f are simple; denote them by $P_1, \dots, P_M$ ($m = 2N + 2g - 2$ according to the Riemann–Hurwitz formula); the critical values $\lambda_k = \pi(P_k)$ can be used as (local) coordinates on $H_{g,N}(k_1, \dots, k_n)$. The function f defines the realization of the Riemann surface $\mathcal{L}$ as an N-sheeted branched covering of $\mathbb{C}P^1$ with ramification points $P_1, \dots, P_m$ and branch points $\lambda_k = f(P_k)$; we denote points at infinity by $\infty_1, \dots, \infty_n$. In a neighborhood of the ramification point P_k the local coordinate is chosen to be $x_k := \sqrt{\lambda - \lambda_k}$, $k = 1, \dots, m$; in a neighborhood of the point ∞_j the local parameter is $x_{m+j} := \lambda^{-1/k_j}$.

The bidifferential $B(P, Q)$ has a second order pole as $Q \to P$ with the local behavior (1.12): $B(P, Q)/\{dz(P)\,dz(Q)\} = (z(P) - z(Q))^{-2} + \frac{1}{6} S_B(P) + o(1)$, where $z(P)$ is a local coordinate; $S_B(z(P))$ is the Bergmann projective connection.

We define the Bergmann τ-function $\tau_f(\lambda_1, \dots, \lambda_m)$ locally by the system of Eqs. (3.23):

$$\frac{\partial}{\partial \lambda_k} \ln \tau_f = -\frac{1}{12} S_B(x_k)\bigg|_{x_k=0}, \quad k = 1, \dots, m, \tag{4.1}$$

compatibility of this system is a simple corollary of the Rauch variational formulas [21].

Consider the divisor of the differential df: $(df) = \sum_{k=1}^{m+n} r_k D_k$ where $D_k := P_k$, $r_k := 1$ for $k = 1, \dots, m$ and $D_{m+j} = \infty_j$, $r_{m+j} = -(k_j + 1)$ for $j = 1, \dots, n$; the corresponding local parameters x_k, $k = 1, \dots, m + n$ were introduced above.

Here and below, if an argument of a differential coincides with a point D_j of the divisor (df), we evaluate this differential at the point D_j with respect to the local parameter x_j. In particular, for the prime form we shall use the following conventions:

$$E(D_k, D_l) := E(P, Q)\sqrt{dx_k(P)}\sqrt{dx_l(Q)}\,\big|_{P=D_k, Q=D_l} \tag{4.2}$$

for $k, l = 1, \dots, m + N$. The next notation corresponds to prime-forms, evaluated at points of divisor (df) with respect to only one argument:

$$E(P, D_l) := E(P, Q)\sqrt{dx_l(Q)}\,\big|_{Q=D_l}, \tag{4.3}$$

$l = 1, \dots, m + n$; in contrast to $E(D_k, D_l)$, which are just scalars, $E(P, D_l)$ are $-1/2$-forms with respect to P.

Denote by $w_1, \dots, w_g$ normalized ($\oint_{a_\alpha} w_\beta = \delta_{\alpha\beta}$) holomorphic differentials on $\mathcal{L}$; $\mathbf{B}_{\alpha\beta} = \oint_{b_\alpha} w_\beta$ is the corresponding matrix of b-periods; $\Theta(z|\mathbf{B})$ is the theta-function.

Choose some initial point $P \in \hat{\mathcal{L}}$ and introduce the vector of Riemann constants K^P and the Abel map $\mathcal{A}_\alpha(Q) = \int_P^Q w_\alpha$. Since all the zeros of the differential df have multiplicity 1, we can always choose the fundamental cell $\hat{\mathcal{L}}$ of the universal covering of $\mathcal{L}$ in such a way that $\mathcal{A}((df)) = -2K^P$ (for an arbitrary choice of the fundamental cell these two vectors coincide only up to an integer combination of the periods of holomorphic differentials), where the Abel map is computed along a path which does not intersect the boundary of $\hat{\mathcal{L}}$.

The key entry of the Bergmann tau-function is the following holomorphic multivalued $(1 - g)g/2$-differential $\mathcal{C}(P)$ on $\mathcal{L}$:

$$\mathcal{C}(P) := \frac{1}{W(P)} \sum_{\alpha_1, \dots, \alpha_g = 1}^{g} \frac{\partial^g \Theta(K^P)}{\partial z_{\alpha_1} \cdots \partial z_{\alpha_g}} w_{\alpha_1}(P) \cdots w_{\alpha_g}(P), \tag{4.4}$$

where

$$W(P) := \det_{1\leqslant\alpha,\beta\leqslant g} \left\| w_\beta^{(\alpha-1)}(P) \right\| \tag{4.5}$$

denotes the Wronskian determinant of holomorphic differentials.

The following theorem is a slight modification of the theorem proved in [20].

Theorem 1. *The Bergmann tau-function* (4.1) *on the Hurwitz space* $H_{g,N}(k_1,\dots,k_n)$ *is given by the following expression*:

$$\tau_f = \mathcal{Q}^{2/3} \prod_{k,l=1,k<l}^{m+n} \left[E(D_k,D_l)\right]^{\frac{r_k r_l}{6}}, \tag{4.6}$$

where the quantity $\mathcal{Q}$ *defined by*

$$\mathcal{Q} = \left[df(P)\right]^{\frac{g-1}{2}} \mathcal{C}(P) \prod_{k=1}^{m+N} \left[E(P,D_k)\right]^{\frac{(1-g)r_k}{2}} \tag{4.7}$$

is independent of the point $P \in \mathcal{L}$.

The proof of this theorem is very similar to [20]. The only technical difference is appearance of higher order poles of the function f.

4.2. Dependence of the Bergmann tau-function on the choice of the projection

Theorem 2. *Let* τ_f *and* τ_g *be Bergmann tau-functions* (4.6) *corresponding to divisors* (df) *and* (dg)*, respectively. Then*

$$\left(\frac{\tau_f}{\tau_g}\right)^{12} = C \frac{(u_{d_1+1})^{1-\frac{1}{d_1}}}{(v_{d_2+1})^{1-\frac{1}{d_2}}} \frac{\prod_k df(Q_k)}{\prod_k dg(P_k)}, \tag{4.8}$$

where

$$C = \frac{d_1^{d_1+3}}{d_2^{d_2+3}} \tag{4.9}$$

is a constant which is independent of moduli parameters.

Proof. As before, we assume that the fundamental cell $\hat{\mathcal{L}}$ is chosen in such a way that $\mathcal{A}((df)) = \mathcal{A}((dg)) = -2K^P$. Introduce the following convenient notation for the divisors (df) and (dg):

$$(df) = \sum_{k=1}^{m_1} P_k - 2\infty_f - (d_2+1)\infty_g := \sum_{k=1}^{m_1+2} r_k D_k, \tag{4.10}$$

$$(dg) = \sum_{k=1}^{m_2} Q_k - 2\infty_g - (d_1+1)\infty_f := \sum_{k=1}^{m_2+2} s_k G_k. \tag{4.11}$$

Since $\deg(df) = \deg(dg) = 2g - 2$, we have $\sum_{k=1}^{m_1+2} r_k = \sum_{k=1}^{m_2+2} s_k = 2g - 2$. Then, according to the expression (4.7) for the Bergmann tau-function, we have

$$(\tau_f)^{12} = \mathcal{C}^8(P)\big[df(P)\big]^{4g-4} \prod_{k,j=1}^{m_1+2} \big\{E(D_k, D_j)\big\}^{2r_k r_j} \prod_{k=1}^{m_1+2} \big\{E(P, D_k)\big\}^{r_k(4-4g)}, \tag{4.12}$$

where the values of all prime-forms at the points of the divisor (df) are evaluated in the system of local parameters defined by the function f: near P_k the local parameter is $x_k(P) = \sqrt{f(P) - \lambda_k}$; near ∞_f the local parameter is $x_{m_1+1}(P) = 1/f(P)$, and near ∞_g the local parameter is $x_{m_1+2} = [f(P)]^{-1/d_2}$.

Similarly, we have

$$(\tau_g)^{12} = \mathcal{C}^8(P)\big[dg(P)\big]^{4g-4} \prod_{k,j=1}^{m_2+2} \big\{E(G_k, G_j)\big\}^{2s_k s_j} \prod_{k=1}^{m_2+2} \big\{E(P, G_k)\big\}^{s_k(4-4g)}, \tag{4.13}$$

where values of all prime-forms at the points of the divisor (dg) should be evaluated in the system of local parameters defined by the function g: near Q_k the local parameter is $y_k(P) = \sqrt{g(P) - \mu_k}$; near ∞_f the local parameter is $y_{m_2+1}(P) = 1/g(P)$, and near ∞_g the local parameter is $y_{m_2+2}(P) = [g(P)]^{-1/d_2}$.

Therefore,

$$\left(\frac{\tau_f}{\tau_g}\right)^{12} = \frac{\prod_{k,j=1}^{m_1+2}\{E(D_k, D_j)\}^{2r_k r_j}}{\prod_{k,j=1}^{m_2+2}\{E(G_k, G_j)\}^{2s_k s_j}} \left\{\frac{df(P)}{dg(P)} \frac{\prod_{k=1}^{m_2+2}\{E(P, G_k)\}^{s_k}}{\prod_{k=1}^{m_1+2}\{E(P, D_k)\}^{r_k}}\right\}^{4g-4}. \tag{4.14}$$

Using independence of this expression of the point P, we can split the $(4g - 4)$th power into the product over points of the divisor $(df) + (dg)$ (the degree of this divisor equals exactly $4g - 4$). It is important to remember that, evaluating the prime-forms and differentials df and dg at the points D_k and G_k we fix the local parameters (these local parameters at the points of (df) are defined via the function f, and at the points of (dg) via the function g as explained above). Since the divisors (df) and (dg) have common points (∞_f and ∞_g), in a neighborhood of each of these points we introduce two essentially different local parameters, and it is important to remember in each case in which local parameter the prime-forms are computed.

Another subtlety is that, being considered as functions of P, different multipliers in (4.14) either vanish or become singular if $P \in (df) + (dg)$; cancellation of these singularities should be accurately traced down.

Consider the first "half" of this expression, namely, the product over $P \in (df)$:

$$\begin{aligned}&\left\{\frac{df(P)}{dg(P)} \frac{\prod_{k=1}^{m_2+2}\{E(P, G_k)\}^{s_k}}{\prod_{k=1}^{m_1+2}\{E(P, D_k)\}^{r_k}}\right\}^{2g-2} \\ &\quad = \prod_{l=1}^{m_1+2} \lim_{P \to D_l} \left\{\frac{df(P)}{dg(P)} \frac{\prod_{k=1}^{m_2+2}\{E(P, G_k)\}^{s_k}}{\prod_{k=1}^{m_1+2}\{E(P, D_k)\}^{r_k}}\right\}^{r_l}\end{aligned} \tag{4.15}$$

$$= \prod_{k,l=1,\,k<l}^{m_1+2} \bigl\{E(D_l, D_k)\bigr\}^{-2r_k r_l} \prod_{k=1}^{m_1+2} \left\{ \lim_{P\to D_k} \frac{df(P)}{\{E(P, D_k)\}^{r_k}} \right\}^{r_k}$$
$$\times \prod_{l=1}^{m_1+2} \left\{ \lim_{P\to D_l} \frac{\prod_{k=1}^{m_2+2}\{E(P, G_k)\}^{s_k}}{dg(P)} \right\}^{r_l}. \tag{4.16}$$

The first product looks nice since it cancels out against the first product in the numerator of (4.14). Let us evaluate other ingredients of this expression. We have $D_k = P_k$, $r_k = 1$ for $k = 1, \ldots, m_1$, $D_{m_1+1} = \infty_f$, $k_{m_1+1} = -2$, $D_{m_1+2} = \infty_g$, $k_{m_1+2} = -(d_2 + 1)$. Therefore,

$$\prod_{k=1}^{m_1+2} \left\{ \lim_{P\to D_k} \frac{df(P)}{\{E(P, D_k)\}^{r_k}} \right\}^{r_k}$$
$$= \left\{ \lim_{P\to D_{m_1+1}} \bigl\{df(P)\, E^2(P, D_{m_1+1})\bigr\} \right\}^{-2}$$
$$\times \left\{ \lim_{P\to D_{m_1+2}} \bigl\{df(P)\, E^{d_2+1}(P, D_{m_1+2})\bigr\} \right\}^{-d_2-1} \prod_{k=1}^{m_1} \lim_{P\to P_k} \frac{df(P)}{\{E(P, P_k)\}}, \tag{4.17}$$

where we do not write ∞_f and ∞_g instead of D_{m_1+1} and D_{m_1+2}, respectively, to indicate that we need to use the system of local parameters related to $f(P)$. The last term in the product (4.17) is the easiest one:

$$\lim_{P\to P_k} \frac{df(P)}{\{E(P, P_k)\}} = \lim_{x_k(P)\to 0} \frac{2x_k}{x_k} = 2. \tag{4.18}$$

In a similar way we evaluate the first term:

$$\lim_{P\to D_{m_1+1}} \bigl\{df(P)\, E^2(P, D_{m_1+1})\bigr\} = -1, \tag{4.19}$$

and the second one:

$$\lim_{P\to D_{m_1+2}} \bigl\{df(P)\, E^{d_2+1}(P, D_{m_1+2})\bigr\} = -d_2. \tag{4.20}$$

It remains to evaluate the third product in (4.16):

$$\prod_{l=1}^{m_1+2} \left\{ \lim_{P\to D_l} \frac{\prod_{k=1}^{m_2+2}\{E(P, G_k)\}^{s_k}}{dg(P)} \right\}^{r_l}$$
$$= \left(\prod_{l=1}^{m_1} \bigl\{dg(P_l)\bigr\}^{-1} \right) \left(\prod_{\text{all } k,l \text{ such that } D_l \neq G_k} \bigl\{E(D_l, G_k)\bigr\}^{r_l s_k} \right)$$
$$\times \left(\lim_{P\to D_{m_1+1}} \bigl\{E(P, G_{m_2+2})\bigr\}^{d_1+1} dg(P) \right)^2$$
$$\times \left(\lim_{P\to D_{m_1+2}} \bigl\{E(P, G_{m_2+1})\bigr\}^2 dg(P) \right)^{d_2+1}. \tag{4.21}$$

Consider the first limit in (4.21):

Lemma 3.

$$\lim_{P\to D_{m_1+1}}\left(\left\{E(P,G_{m_2+2})\right\}^{d_1+1}dg(P)\right)^2=\left(d_1^2\right)(u_{d_1+1})^{1-\frac{1}{d_1}}. \tag{4.22}$$

Proof. Two different local parameters at the point $\infty_f\equiv D_{m_1+1}\equiv G_{m_2+2}$ which we need to use are $x_{m_1+1}(P)=f^{-1}(P)$ and $y_{m_2+2}(P)=g^{-1/d_1}(P)$. We have

$$E(P,G_{m_2+2})=\frac{(y_{m_2+2}(P)+\cdots)}{d\sqrt{y_{m_2+2}(P)}}=\sqrt{\frac{dx_{m_1+1}}{dy_{m_2+2}}(\infty_f)}\frac{(y_{m_2+2}(P)+\cdots)}{\sqrt{dx_{m_1+1}(P)}}. \tag{4.23}$$

Taking into account that $g(P)=y_{m_2+2}^{-d_1}$, we have

$$dg(P)=-(d_1)(y_{m_2+2})^{-d_1-1}\left(\frac{dy_{m_2+2}}{dx_{m_1+1}}(\infty_f)\right)dx_{m_1+1}(P). \tag{4.24}$$

Taking in (4.22) the limit $P\to D_{m_1+1}$, we indicate that all differentials in the bracket should be evaluated with respect to the local parameter x_{m_1+1}. Therefore, in (4.22) we ignore all factors $dx_{m_1+1}(P)$; then (4.22) turns out to be equal to

$$\left(d_1^2\right)\left(\frac{dy_{m_2+2}}{dx_{m_1+1}}(\infty_f)\right)^{1-d_1}=\left(d_1^2\right)(u_{d_1+1})^{1-\frac{1}{d_1}}, \tag{4.25}$$

where we took into account that, as $P\to\infty_f$, $g=u_{d_1+1}x^{d_1}+\cdots$; thus $(dy_{m_2+2}/dx_{m_1+1})(\infty_f)=(u_{d_1+1})^{-1/d_1}$. $\square$

Consider now the second limit in (4.21):

Lemma 4.

$$\lim_{P\to D_{m_1+2}}\left\{E(P,G_{m_2+1})\right\}^2dg(P)=-1. \tag{4.26}$$

Proof. In analogy to (4.22) we have to evaluate the prime-form and the differential dg in the local parameter related to the function f, which is given by $x_{m_1+2}(P)=(f(P))^{-1/d_2}$ (the local parameter near this point related to the function g is $y_{m_2+1}(P)=(g(P))^{-1}$). We have near D_{m_1+2}:

$$E(P,G_{m_2+1})=\frac{y_{m_2+1}(P)+\cdots}{\sqrt{y_{m_2+1}(P)}}=\sqrt{\frac{x_{m_1+2}}{dy_{m_2+1}}(\infty_g)}\frac{y_{m_2+1}(P)+\cdots}{\sqrt{dx_{m_1+2}(P)}}, \tag{4.27}$$

and

$$dg(P)=d\left(\frac{1}{y_{m_2+1}(P)}\right)=-\frac{dy_{m_2+1}}{x_{m_1+2}}(\infty_g)\frac{dx_{m_1+2}(P)}{y_{m_2+1}^2(P)}. \tag{4.28}$$

As before, substituting these expressions to (4.26) and ignoring the arising power of $dx_{m_1+2}(P)$, we see that this limit equals -1. $\square$

Substituting this (4.26), (4.25) and (4.22) into (4.21), and collecting all terms in (4.16), we get

$$\left\{\frac{df(P)}{dg(P)}\frac{\prod_{k=1}^{m_2+2}\{E(P,G_k)\}^{s_k}}{\prod_{k=1}^{m_1+2}\{E(P,D_k)\}^{r_k}}\right\}^{2g-2}$$
$$= \left\{2d_1^2 d_2^{-(d_2+1)}\right\}(u_{d_1+1})^{1-\frac{1}{d_1}}\left(\prod_{l=1}^{m_1}\left\{dg(P_l)\right\}^{-1}\right)$$
$$\times \frac{\prod_{D_l\neq G_k}\{E(D_l,G_k)\}^{r_l s_k}}{\prod_{k,l=1,\,k<l}^{m_1+2}\{E(D_l,D_k)\}^{2r_k r_l}}. \tag{4.29}$$

Now, computing the second "half" of (4.14), i.e., taking the product analogous to (4.16) over points of divisor (dg), and multiplying it by (4.29), we come to the statement of Theorem 2. □

4.3. Bergmann tau-function and F^1

Theorem 3. *The solution F^1 Eqs. (3.1), (3.2), (3.20) is given by the following equivalent formulas*:

$$F^1 = \frac{1}{24}\ln\left\{\tau_f^{12}(v_{d_2+1})^{1-\frac{1}{d_2}}\prod_{k=1}^{m_1} dg(P_k)\right\} + \frac{d_2+3}{24}\ln d_2 + C, \tag{4.30}$$

or

$$F^1 = \frac{1}{24}\ln\left\{\tau_g^{12}(u_{d_1+1})^{1-\frac{1}{d_1}}\prod_{k=1}^{m_2} df(Q_k)\right\} + \frac{d_1+3}{24}\ln d_1 + C. \tag{4.31}$$

Here τ_f and τ_g are the Bergmann tau-functions (4.6) built from divisors (df) and (dg), respectively; *C is a constant.*

Proof. From formulas (4.8), (4.9) it follows that expressions (4.30) and (4.31) coincide. According to Proposition 1, the expression (4.30) satisfies Eqs. (3.1), (3.20) with respect to coefficients of V_1. Similarly, the expression (4.31) satisfies the analogous system (3.2) with respect to coefficients of V_2. □

Remark 2 (*Higher order branch points*). If potentials V_1 and V_2 are non-generic, i.e., some (or all) of the branch points have multiplicity higher than 1, the formula (4.31) should be only slightly modified. Namely, the expression for Bergmann tau-function (4.6) formally remains the same in terms of the divisor of the differential df (the zeros of df can now have arbitrary multiplicities). The expression for F^1 then looks as follows:

$$F^1 = \frac{1}{48}\ln\left\{\tau_f^{24}(v_{d_2+1})^{2-\frac{2}{d_2}}\prod_{k=1}^{m_1}\operatorname{res}|_{P_m}\frac{(dg)^2}{df}\right\} + \frac{d_2+3}{24}\ln d_2 + C. \tag{4.32}$$

The proof of (4.32) is slightly more technically involved than the proof in the generic case and will be published separately.

5. Equations with respect to filling fractions

It is well known (see, for example, [17]) that normalized ($\oint_{a_\alpha} w_\beta = \delta_{ab}$) holomorphic differentials can be expressed as follows:

$$2\pi i w_\alpha(P) = \frac{\partial g(P)}{\partial \epsilon_\alpha}\bigg|_{f(P)} df(P). \tag{5.1}$$

(Sketch of the proof: differentiating (2.14) with respect to ϵ_β, we verify the normalization conditions for differentials (5.1). The 1-form $g\,df$ is singular at ∞_f and ∞_g; at ∞_f we have $g = V_1'(f) - 1/f + \cdots$; this singularity disappears since coefficients of V_1 and V_2 are independent of filling fractions. Singularities of the derivative $\partial g/\partial\epsilon_\alpha$ at the branch points P_k are canceled by zeros of df at the same points. At ∞_g we have: $x = V_2'(g) - 1/g + \cdots$; due to the thermodynamic identity

$$\frac{\partial g}{\partial \epsilon_\alpha}\bigg|_f df = -\frac{\partial f}{\partial \epsilon_\alpha}\bigg|_g dg.$$

Since coefficients of V_2 are independent of ϵ_α, the singularity of $g\,df$ at ∞_g also disappears after differentiation.)

To obtain equations for derivatives of F^1 with respect to filling fractions we shall prove the following analog of Lemma 1:

Lemma 5. *The following deformation equations with respect to filling fractions take place*:

$$\partial_{\epsilon_\alpha}\lambda_k = -2\pi i \frac{w_\alpha(P_k)}{g'(P_k)}, \tag{5.2}$$

$$\frac{\partial\{g'(P_k)\}}{\partial\epsilon_\alpha} = \frac{\pi i}{2}\left\{w_a''(P_k) - \frac{g'''(P_k)}{g'(P_k)} w_\alpha(P_k)\right\}. \tag{5.3}$$

The proof is parallel to the proof of (2.16) and (2.17): from (5.1) we have

$$\frac{\partial g(P)}{\partial \epsilon_\alpha}\bigg|_{x_k(P)} df(P) - \frac{\partial f(P)}{\partial \epsilon_\alpha}\bigg|_{x_k(P)} dg(P) = 2\pi i w_\alpha(P). \tag{5.4}$$

Substituting the local expansions (2.22) of $g(P)$ and (2.23) of $dg(P)$, together with the Taylor expansion of $w_\alpha(P)$

$$w_\alpha(P) = \left(w_\alpha(P_k) + w_\alpha'(P_k)x_k + \frac{w_\alpha''(P_k)}{2}x_k^2 + \cdots\right) dx_k, \tag{5.5}$$

into (5.4), we get, since $f(P) = x_k^2(P) + \lambda_k$ and $df(P) = 2x_k(P)\,dx_k(P)$:

$$\begin{aligned}
&\left(\partial_{\epsilon_\alpha} g(P_k) + x_k\partial_{\epsilon_\alpha} g'(P_k) + \frac{1}{2}\partial_{\epsilon_\alpha} g''(P_k) + \cdots\right) 2x_k\,dx_k \\
&\quad - \partial_{\epsilon_\alpha} f_k\left(g'(P_k) + g''(P_k)x_k + \frac{1}{2}g'''(P_k)x_k^2 + \cdots\right) dx_k \\
&= 2\pi i\left(w_\alpha(P_k) + w_\alpha'(P_k)x_k + \frac{1}{2}w_\alpha''(P_k)x_k^2\right) dx_k.
\end{aligned}$$

The zeroth order term gives (5.2). Collecting coefficients in front of x_k^2, and using (5.2), we get (5.3).

Theorem 4. *Derivatives of the function* F^1 (4.30), (4.31) *with respect to filling fractions look as follows*:

$$\frac{\partial F^1}{\partial \epsilon_\alpha} = -\oint_{b_\alpha} Y^1(P)\, df(P), \tag{5.6}$$

where $Y^1 df$ *is defined by* (3.20).

Proof. The vectors of b-periods of 1-forms $B(P, P_k)$ and $D(P, P_k)$ can be expressed in terms of the holomorphic differentials via the following standard formulas:

$$\oint_{b_a} B(P, P_k) = 2\pi i w_\alpha(P_k), \qquad \oint_{b_\alpha} D(P, P_k) = 2\pi i w''_\alpha(P_k). \tag{5.7}$$

Therefore, the b-periods of the 1-form $-Y^{(1)}(P)\, df(P)$ defined by (3.20) are given by the following expression:

$$\begin{aligned} &-\oint_{b_\alpha} Y^{(1)}(P)\, df(P) \\ &\quad = 2\pi i \sum_{k=1}^{m_1} \left\{ -\frac{w''_a(P_k)}{96 g'(P_k)} + \frac{g'''(P_k) w_a(P_k)}{96 g'^2(P_k)} + \frac{S_B(P_k) w_a(P_k)}{24 g'(P_k)} \right\}. \end{aligned} \tag{5.8}$$

On the other hand, derivatives of F^1 (4.30) with respect to ϵ_α can be computed using (5.2), (5.3) and equations for the Bergmann tau-function (3.23); the result coincides with (5.8). □

6. F^1 and related objects

6.1. F^1, isomonodromic tau-function and G-function of Frobenius manifolds

We recall that the genus 1 correction to free energy in topological field theories is given by the so-called G-function of the associated Frobenius manifolds. The G-function is a solution of the Getzler equation [29]; for Frobenius manifolds related to quantum cohomologies, the G-function was intensively studied as a generating function of elliptic Gromov–Witten invariants (see [24,30] for references). In [24] it was found the following formula for the G-function of an arbitrary m-dimensional semisimple Frobenius manifold:

$$G = \ln \frac{\tau_I}{\prod_{k=1}^m \eta_{kk}^{1/48}}, \tag{6.1}$$

where τ_I is the Jimbo–Miwa tau-function of Riemann–Hilbert problem associated to a given Frobenius manifold [18]; η_{kk} are the coefficients of the Darboux–Egoroff (pseudo)metric corresponding to the semisimple Frobenius manifold.

One of the well-studied classes of Frobenius manifolds arises from Hurwitz spaces [18]. For these Frobenius manifolds the isomonodromic tau-function τ_I [18] is related to the Bergmann tau-function τ_f (3.23) as follows [25]:

$$\tau_I = \tau_f^{-1/2}. \tag{6.2}$$

Therefore, the tau-function term is the same in the formulas (4.30) for F^1 and (6.1) for the G-function (up to a sign, which is related to the choice of the sign in the exponent in the definition (1.1) of the free energy). The solution of the Fuchsian system corresponding to the tau-function τ_I is not known explicitly. However, the same function τ_I, being multiplied with a theta-functional factor, equals the tau-function of a Riemann–Hilbert problem with quasi-permutation monodromy matrices which was solved in [26].

The metric coefficients of the Darboux–Egoroff metric, corresponding to a Hurwitz Frobenius manifold, are given in terms of an "admissible" 1-form φ, defining the Frobenius manifold:

$$\eta_{kk} = \mathrm{res}|_{P_k} \frac{\varphi^2}{df}. \tag{6.3}$$

If, trying to develop an analogy with our formula (4.30) for F^1, we formally choose $\phi(P) = dg(P)$, we get $\eta_{kk} = g'^2(P_k)/2$ and the formula (6.1) coincides with (4.30) up to small details like sign, additive constant and the highest coefficient of the polynomial V_2 arising from the requirement of symmetry between f and g.

Therefore, we observe a formal analogy between our expression (4.30) for F^1 and the Dubrovin–Zhang formula (6.1) for the G-function. Unfortunately, at the moment this analogy remains only formal, since, from the point of view of Dubrovin's theory [18], the differential dg is not admissible; therefore, the metric $\eta_{kk} = g'^2(P_k)/2$ built from this differential is not flat; thus it does not define a Frobenius manifold. Therefore, the true origin of the analogy between the G-function of Frobenius manifolds and F^1 still has to be explored.

6.2. F^1 and determinant of Laplace operator

Existence of a close relationship between F^1 and the determinant of certain Laplace operator was suggested by several authors (see, e.g., [27] for Hermitian one-matrix model, [15] for Hermitian two-matrix model and, finally, [28] for normal two-matrix model with simply-connected support of eigenvalues). In particular, in [28] F^1 was claimed to coincide with the determinant of Laplace operator acting on functions satisfying Dirichlet boundary conditions in some domain.

However, in the context of Hermitian two-matrix model (as well as in the case of Hermitian one-matrix model [27]) this relationship is more subtle.

First, if we do not impose any reality conditions on coefficients of polynomials V_1 and V_2, the function F^1 is a holomorphic function of the moduli parameters (i.e., coefficients of V_1, V_2 and filling fractions), while $\det \Delta$ is always a real-valued function. The Laplace operator Δ^f which should be playing a role here corresponds to the singular metric $|df|^2$ of infinite volume.

This problem disappears if we start from more physical situation, when all moduli parameters are real, as well as the branch points of the Riemann surface $\mathcal{L}$ with respect to both projections. In this case F^1 is real-valued itself, as well as the determinant of the Laplace operator. However, little is known about rigorous definition of the determinants of Laplace operators for the infinite volume, although such determinants were actively used by string theorists without mathematical justification [31–33]. According to empirical results of [33], the regularized determinant of Laplace operator Δ^f is given by the formula

$$\frac{\det \Delta^f}{\mathrm{Vol}(\mathcal{L}) \det \Im \mathbf{B}} = C|\tau_f|^2, \tag{6.4}$$

where $\mathrm{Vol}(\mathcal{L})$ is a regularized area of $\mathcal{L}$, Δ^f is the Laplace operator defined in the singular metric $|df(P)|^2$, $\mathbf{B}$ is the matrix of b-periods of $\mathcal{L}$, C is a constant.

In the "physical" case of real moduli parameters the empirical expression (6.4) for $\ln\{\det \Delta^f\}$ coincides with F^1 (4.30) up to a simple power and additional multipliers.

Therefore, the relationship between Hermitian and normal two-matrix models [28] on the level of F^1 is not as straightforward as on the level of the functions F^0 (F^0 for Hermitian two-matrix model can be obtained from F^0 for normal two-matrix model by a simple analytical continuation [16,17,22,34]).

From the formula (6.4) we see that Theorem 2 which describes the dependence of the Bergmann tau-function on the projection choice is nothing but a version (working for flat singular metrics) of Alvarez–Polyakov formula [35], which describes the change of $\det \Delta$ if the metric changes within a given conformal class.

7. Partial cases

7.1. From two-matrix to one-matrix model: hyperelliptic curves

Suppose that $d_2 = 1$, i.e., the polynomial V_2 is quadratic. Then the integration with respect to M_2 in (1.1) can be carried out explicitly, and we get the partition function of the one-matrix model:

$$Z_N \equiv e^{-N^2 F} = C \int dM\, e^{-N \operatorname{tr} V(M)}, \tag{7.1}$$

where $M := M_1$, $V := V_1$ and C is a constant.

For $d_2 = 1$ the function $f(P)$ has two poles of order 1 at ∞_f and ∞_g; thus, the spectral curve $\mathcal{L}$ is hyperelliptic and the function $f(P)$ defines a two-sheeted branched covering of the Riemann sphere. The number of branch points in this case equals $m_1 \equiv 2g + 2$; as before, we call them $\lambda_1, \ldots, \lambda_{2g+2}$. The Bergmann tau-function (3.23) for hyperelliptic curves was computed in [23]; in this case it admits the following, alternative to (4.6), (4.7), expression:

$$\tau_f = \Delta^{1/4} \det \mathbf{A}, \tag{7.2}$$

where

$$\Delta := \prod_{j<k,\, j,k=1}^{2g+2} (\lambda_j - \lambda_k); \tag{7.3}$$

$\mathbf{A}$ is the matrix of a-periods of non-normalized holomorphic differentials on $\mathcal{L}$:

$$\mathbf{A}_{\alpha\beta} = \oint_{a_\alpha} \frac{x^{\beta-1}\, dx}{\nu}. \tag{7.4}$$

Here

$$\nu^2 = \prod_{k=1}^{2g=2} (x - \lambda_k)$$

is the equation of the spectral curve $\mathcal{L}$.

Substituting the formula (7.2) into (4.30), and ignoring the coefficient v_{d_2+1} (it becomes a part of the constant C), we get the expression

$$F^1 = \frac{1}{24} \ln \left\{ \Delta^3 (\det \mathbf{A})^{12} \prod_{k=1}^{2g+2} g'(\lambda_k) \right\}, \tag{7.5}$$

which agrees with previously known results [9–11,13].

7.2. Rational spectral curve (“one-cut” case)

For the “one-cut” case, when the spectral curve $\mathcal{L}$ has genus zero, F^1 was computed in [14]. This result can be deduced from our present formalism as follows. For genus zero the expression for the Bergmann tau-function (1.22) can be rewritten in terms of the uniformization map $z(P)$ of the Riemann surface $\mathcal{L}$ to the Riemann sphere, satisfying the condition $z(P) = \lambda + O(1)$ as $P \to \infty_f$. The formula for τ_f looks as follows (see (3.32), (4.5) in [25]):

$$\tau_f^{12} = (v_{d_2+1})^{1+1/d_2} \prod_{k=1}^{d_2+1} \frac{dx_k}{dz}(P_k).$$

This expression can be derived from (1.22) using the formula for the prime-form on $\mathcal{L}$ obtained as pull-back of the prime-form on the Riemann sphere:

$$E(P,Q) = \frac{z(P) - z(Q)}{\sqrt{dz(P)}\sqrt{dz(Q)}}.$$

Substituting this formula into (1.17) and using the chain rule $\frac{dg}{dx_k}(P_k)\frac{dx_k}{dz}(P_k) = \frac{dg}{dz}(P_k)$, we rewrite (1.17) as follows:

$$F^1 = \frac{1}{24}\ln\left\{v_{d_2+1}^2 \prod_{k=1}^{d_2+1} \frac{dg}{dz}(P_k)\right\} + C,$$

where C is a constant, in agreement with the formula previously obtained in [14].

7.3. *Elliptic spectral curve ("two-cut" case)*

Denote the period of the spectral curve $\mathcal{L}$ by σ. The Bergmann tau-function (1.22) for elliptic covering with multiplicities of points at infinity equal to 1 and d_2 can be represented as follows ([25], (3.35)):

$$\tau_f^{12} = \eta^{24}(\sigma)\left(\frac{w}{d(f^{-1})}(\infty_f)\right)^2\left(\frac{w}{d(f^{-1/d_2})}(\infty_g)\right)^{d_2+1} \prod_{k=1}^{d_2+3} \frac{dx_k}{w}(P_k), \tag{7.6}$$

where $\eta(\sigma) = [\vartheta_1'(0,\sigma)]^{1/3}$ is the Dedekind eta-function; w is an arbitrary holomorphic one-form on $\mathcal{L}$ (it is easy to see that (7.6) remains invariant if w is multiplied by an arbitrary constant). For simplicity we can normalize w such that at ∞_g we get $w(P) = d(f^{-1/d_2}(P))[1 + o(1)]$. Under this normalization we get the following expression for F^1:

$$F^1 = \ln\eta(\sigma) + \frac{1}{24}\ln\left\{(v_{d_2+1})^{1+1/d_2}\left(\frac{w}{d(f^{-1})}(\infty_f)\right)^2 \prod_{k=1}^{d_2+3} \frac{dg}{w}(P_k)\right\} + C, \tag{7.7}$$

which is new; it looks different (although defines the same function) from the expression previously obtained in [15]. The expression obtained in [15] can be derived by straightforward specialization of the formula (1.17) to genus 1 case using the following expression for the prime-form in genus one:

$$E(P,Q) = \frac{\vartheta_1'(z(P) - z(Q))}{\vartheta_1'(0)\sqrt{dz(P)}\sqrt{dz(Q)}},$$

where $z(P)$ is the uniformization map of the curve $\mathcal{L}$ to the torus with periods 1 and σ (in the elliptic case the differential $\mathcal{C}(P)$ does not depend on P and equals $\vartheta_1'(0)$).

Acknowledgements

We thank M. Bertola, L. Chekhov, B. Dubrovin, T. Grava, V. Kazakov, I. Kostov, M. Staudacher and S. Theisen for important discussions. The work of B.E. was partially supported by the EC ITH Network HPRN-CT-1999-000161. The work of D.K. was partially supported by NSERC, NATEQ and Humboldt foundation. A.K. and D.K. thank Max-Planck Institute for Mathematics in Bonn for support and nice working conditions. D.K. thanks also SISSA and CEA for support and hospitality. B.E. thanks CRM for support and hospitality.

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