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THE NEUMANN PROBLEM FOR THE WAVE EQUATION IN A CONE

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The Neumann problem for the wave equation in a wedge is considered. The asymptotic behavior of solutions to the problem in a neighborhood of the edge of the wedge is studied. In order to deduce and justify asymptotic formulas, the solvability of the problem in the scale of weight function spaces is investigated. Bibliography: 30 *titles.*

Let K be an open cone in $\mathbb{R}^{n-d}$ with vertex at the origin. We suppose that K is smooth outside the vertex. Let $0 \leqslant d \leqslant n-2$, and let $\mathcal{K} = K \times \mathbb{R}^d = \{x = (y,z) \in \mathbb{R}^n : y \in K, z \in \mathbb{R}^d\}$ be a wedge in $\mathbb{R}^n$. In the cylinder

$$Q = \{(x,t) : x \in \mathcal{K}, -\infty < t < +\infty\},$$

we consider the problem

$$\begin{gathered} \Box\, u(x,t) := (\partial_t^2 - \Delta_x)u(x,t) = f(x,t), \quad (x,t) \in Q, \\ \partial_\nu u(x,t) = 0, \quad (x,t) \in \partial Q, \end{gathered} \tag{0.1}$$

where ν is the unit outward normal to $\partial\mathcal{K}$.

The main goal of this paper is to study the asymptotic behavior of solutions to the problem (0.1) in a neighborhood of the edge of the wedge $\mathcal{K}$. To derive and justify asymptotic formulas, we study the solvabitliy of the problem (0.1) in the scale of weight function spaces.

The Dirichlet problem was studied for the wave equation in Q (cf. [1]) for strongly hyperbolic systems of second-order differential equations (cf. [2]). The methods of the above-mentioned papers were based on "combined" estimates for solutions. In this paper, we develop this approach for the Neumann problem (0.1).

We explain briefly what we mean by combined estimates. We set

$$L(D_y, D_z, D_t) = \Box.$$

We apply the Fourier transform $F_{(z,t)\to(\zeta,\tau)}$ to the problem (0.1), where $\zeta \in \mathbb{R}^d$, $\tau = \sigma - i\gamma$, $\sigma \in \mathbb{R}$, $\gamma > 0$. Then the Neumann problem for the operator $L(D_y, \zeta, \tau)$ in the cone K appears. This operator is elliptic for fixed parameters ζ and τ. However, the dependence on τ is "hyperbolic." It is required to estimate solutions uniformly with respect to the parameters. To this end the cone K is divided into zones. In a neighborhood of the vertex, in the zone

$$\{y \in K : |y| < c_0 p^{-1}, p = (|\zeta|^2 + |\tau|^2)^{1/2}\},$$

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the known weight elliptic estimate for solutions is used. Far from the vertex, for $|y| > c_1|p|^{-1}$, we prove a weight hyperbolic estimate. Finally, in the intermediate zone, we use a weak global estimate in the entire cone which follows from the well-known estimate

$$\gamma^2 \int\limits_Q \exp(-2\gamma t)|\nabla_{x,t} u(x,t)|^2 \, dx\, dt \leqslant c \int\limits_Q \exp(-2\gamma t)|\Box u(x,t)|^2 \, dx\, dt \tag{0.2}$$

for the problem in the cylinder Q with the homogeneous boundary condition. Thus, we obtain an a priori combined estimate for solutions in the scale of weight function spaces. The following inequality is an example of such an estimate:

$$\|\chi_p v; H^2_\beta(K;p)\|^2 + \gamma^2 \|v; H^1_\beta(K;p)\|^2 \leqslant c \int\limits_K |L(D_y, \zeta, \tau)v|^2 (|y|^{2\beta} + p^{2(1-\beta)}/\gamma^2)\, dy, \tag{0.3}$$

where K is a cone of dimension $n-d>2$, v satisfies the homogeneous Neumann condition on ∂K, $\chi_p(y) = \chi(py)$, χ is a cut-off function such that $\chi(y) = 1$ for $|y| < 1$ and $\chi(y) = 0$ for $|y| > 2$, the norm in $H^s_\beta(K;p)$ is defined by formula (1.3) below. For β we can take any number of the interval $(-\infty, 1]$ except for some set of isolated values. The constant c in (0.3) is independent of v and the parameters $\zeta \in \mathbb{R}^d$, $\tau = \sigma - i\gamma$, $\sigma \in \mathbb{R}$, $\gamma > 0$.

In the case $n-d>2$, the method of [1] can be applied to the problem (0.1) after a slight modification. The case $n-d=2$ should be considered separately. As in the case of elliptic problems, the weight norms should be changed (cf. [3–5], where the Neumann elliptic problem was considered in domains with edges).

Owing to combined estimates, it is possible to study the solvability of the problem in a cone for the operator $L(D_y, \zeta, \tau)$ in the scale of weight spaces and, after that, derive and justify asymptotic formulas for solutions. Under suitable conditions on the right-hand side, such formulas consist of linear combinations of "asymptotic" solutions to the corresponding homogeneous problem in the cone. The coefficients of these combinations are explicitly expressed in terms of exact solutions (growing in a neighborhood of the vertex) to the adjoint homogeneous problem. Using the inverse Fourier transform $F^{-1}_{(\zeta,\tau)\to(z,t)}$, we expand these results to the problem in the cylinder Q.

This scheme of the study of the problem (0.1) can be also used in the study of some class of hyperbolic systems that includes the dynamical system of equations of elasticity theory. The authors intend to devote another paper to this question.

Furthermore, we consider formulas for the coefficients in the asymptotic expansion in detail. Solutions to the homogeneous problem occurring in the formulas for the coefficients can be expressed in terms of a hypergeometric function. Therefore, in terms of the coefficients in the asymptotic expansions of solutions to the problems in the cylinder Q, we can express the effects of finite propagation speed for disturbances as follows: the coefficients vanish before the leading edge of the disturbance arrives (which is clear from the general point of view) and the coefficients become infinitely smooth functions of the time variable after the trailing edge passes (which follows from the formulas obtained). Similar questions for the Dirichlet problem were considered in [6].

We indicate some publications devoted to hyperbolic problems in domains with singularities. The methods and results of these publications are different from those presented in this paper. Problems for the wave equation in a wedge with edge of codimension 2 was studied by Eskin in [7], where the homogeneous differential operators of any order with constant coefficients were given on the sides of the wedge. These operators satisfy the uniform Lopatinskii condition. The main result of [7] is an explicit formula for solutions. The method (the reduction to the Riemann–Hilbert problem) is not generalized to the case of a wedge with edge of codimension larger than 2. The asymptotic behavior of solutions was not discussed in [7].

The approach used in the paper [8], devoted to the Cauchy–Dirichlet problem in a domain with conical points, was proposed in [9] in order to describe the asymptotic behavior of solutions in a neighborhood of singularities. In [10], one can find some information about the asymptotic behavior of solutions in a neighborhood of a conical point (in the case of the second-order hyperbolic equation, the Cauchy–Dirichlet problem). In this paper, we use, in fact, the same method as in [9]. The wave equation was considered by Cheeger and Taylor [11] (their approach is based on the functional calculus for the Laplace operator), Uchida [12], and Gerard and Lebeau [13] (microlocal analysis). We also mention earlier works of Kupka and Osher [14], Osher [15], Reisman [16], Sarason [17], and Sarason and Smoller [18].

Section 1 contains some preliminary information. In Sec. 2, we give combined estimates for solutions to the problem in a cone of dimension $n-d>2$. We give only a brief description of these estimates because the situation is close to that considered in [1, 2]. The case $n-d=2$ is discussed in Sec. 3. The properties of the operator of the problem in a cone in the scale of weight spaces are studied in Secs. 4 and 5. The asymptotic behavior of solutions to the problem in a cone is considered in Sec. 6. Section 7 deals with the problem in a wedge. In Sec. 8, we derive explicit formulas for the coefficients of the asymptotic expansions of solutions to the problem (0.1).

§ 1. Preliminaries

1.1. Pencil $\mathfrak{A}(\lambda)$. In a domain $\Omega = K \cap S^{n-d-1}$ on the sphere S^{n-d-1}, we define the operator pencil

$$\mathfrak{A}(\lambda) = (i\lambda)^2 + (n-d-2)i\lambda - \delta \tag{1.1}$$

on functions $u \in H^2(\Omega)$ such that $\partial_\nu u|_{\partial\Omega} = 0$, where ν is the unit outward normal to $\partial\Omega$ and δ is the Laplace–Beltrami operator on S^{n-d-1}. The spectrum of the pencil $\mathfrak{A}(\lambda)$ consists of the normal eigenvalues

$$\lambda_{\pm k} = (i/2)\{(n-d-2) \mp ((n-d-2)^2 + 4\mu_k)^{1/2}\}, \quad k = 0, 1, 2, \dots,$$

where $\{\mu_k\}$ $(0 = \mu_0 < \mu_1 \leqslant \dots)$ is the sequence of all eigenvalues of the operator δ enumerated with repetitions according to multiplicity. With the eigenvalues $\lambda_{\pm k}$ we associate the eigenfunctions Φ_k of the pencil $\mathfrak{A}(\lambda)$ such that

$$\sqrt{(n-d-2)^2 + 4\mu_j}(\Phi_j, \Phi_k)_{L_2(\Omega)} = \delta_k^j.$$

For $n-d>2$ there are no associated functions. In the case $n-d=2$, the pencil $\mathfrak{A}(\lambda)$ has the simple eigenvalues $\lambda_{\pm j} = \mp j(\pi/\alpha)$, $i, j = 1, 2, \dots$, and the double eigenvalue $\lambda_0 = 0$ (α is the angle of the corner K in the plane $\mathbb{R}^2$). The eigenfunction $\Phi_j(\omega)$ corresponding to $\lambda_{\pm j}$, $j > 0$, has the form

$$\Phi_j(\omega) = (j\pi)^{-1/2} \cos(j\pi\omega/\alpha).$$

The eigenfunction $\Phi_0(\omega) = \alpha^{-1/2}$ and the associated function $\Phi_{01}(\omega) \equiv 0$ correspond to the eigenvalue $\lambda_0 = 0$.

1.2. The function space. Let s be a nonnegative integer, and let $\beta \in \mathbb{R}$. We denote by $H^s_\beta(K)$ the completion of $C^\infty_c(\overline{K} \setminus O)$ in the norm

$$\|u; H^s_\beta(K)\| = \Big(\sum_{|\alpha| \leqslant s} \int_K |y|^{2(\beta - s + |\alpha|)} |D^\alpha_y u(y)|^2 \, dy \Big)^{1/2}. \tag{1.2}$$

The space $H^s_\beta(K;q)$, where q is a positive parameter, is equipped with the norm

$$\|u; H^s_\beta(K;q)\| = \Big(\sum_{k=0}^{s} q^{2k} \|u; H^{s-k}_\beta(K)\|^2 \Big)^{1/2}. \tag{1.3}$$

A point x of the wedge $\mathcal{K} = K \times \mathbb{R}^d$ will be written in the form $x = (y, z)$, where $y \in K$ and $z \in \mathbb{R}^d$. We denote by M the edge $O \times \mathbb{R}^d$ of the wedge $\mathcal{K}$, and by Q the cylinder $\{(x,t) : x \in \mathcal{K}, t \in \mathbb{R}\}$. The space $H^s_\beta(Q)$ is the completion of $C^\infty_c(\overline{\mathcal{K}} \setminus M) \times \mathbb{R})$ in the norm

$$\|w; H^s_\beta(Q)\| = \Big(\sum_{|\alpha| \leqslant s} \int_{\mathcal{K}} \int_{\mathbb{R}} |y|^{2(\beta - s + |\alpha|)} |D^\alpha_{x,t} w(x,t)|^2 \, dx\, dt \Big)^{1/2}. \tag{1.4}$$

The $H^s_\beta(Q;q)$-norm is given by formula (1.3) with Q instead of K. We denote by $V^s_\beta(Q;\gamma)$, $\gamma > 0$, the space equipped with the norm

$$\|w; V^s_\beta(Q;\gamma)\| = \|w^\gamma; H^s_\beta(Q;\gamma)\|, \tag{1.5}$$

where

$$w^\gamma(x,t) = \exp(-\gamma t) w(x,t).$$

Let

$$\widehat{w}(y, \zeta, \tau) = F_{(z,t) \to (\zeta,\tau)} w(y,z,t) = \int \exp(-i\zeta z - i\tau t) w(y,z,t) \, dz\, dt,$$

where $\tau = \sigma - i\gamma$, $\sigma \in \mathbb{R}$. We set

$$p = p(\zeta, \tau) = (|\zeta|^2 + |\tau|^2)^{1/2}, \quad \eta = py, \quad W(\eta, \zeta, \tau) = \widehat{w}(p^{-1}\eta, \zeta, \tau). \tag{1.6}$$

We can check (cf. [1]) that the norm $\|w; V^\beta_s(Q, \gamma)\|$ is equivalent to each of the following norms:

$$\begin{gathered} \Big(\int \|\widehat{w}(\,, \zeta, \tau); H^s_\beta(K;p)\|^2 \, d\zeta\, d\sigma \Big)^{1/2}, \\ \Big(\int p^{d-n-2(\beta - s)} \|W(\,, \zeta, \tau); H^s_\beta(K, 1)\|^2 \, d\zeta\, d\sigma \Big)^{1/2} \end{gathered} \tag{1.7}$$

(the constants in the corresponding equivalence relations are independent of $\gamma > 0$).

1.3. Energy estimate for solutions to the problem (0.1), (0.2). In fact, the following assertion is well known.

Proposition 1.1. *Let* $u \in S(\mathbb{R}^n)$, $\partial_\nu u(x,t) = 0$, $(x,t) \in \partial Q$. *Then for any* $\gamma > 0$ *we have*

$$\gamma^2 \int_{-\infty}^{+\infty} \exp(-2\gamma t) \|\nabla_{x,t} u(,t); L_2(\mathcal{K})\|^2 \, dt \leqslant c \int_{-\infty}^{+\infty} \exp(-2\gamma t) \|(\Box u)(,t); L_2(\mathcal{K})\|^2 \, dt, \tag{1.8}$$

where the constant c *is independent of* u *and* $\gamma > 0$.

In the case of the Dirichlet boundary condition, the proof of the estimate (1.8) can be found, for example, in [2, Proposition 1.1]. The same arguments are suitable for the Neumann condition.

§ 2. Combined Estimate for Solutions to the Problem in a Cone K. Case $n - d > 2$

2.1. Statement of the problem in a cone. Applying the Fourier transform $F_{(z,t)\to(\zeta,\tau)}$ to the problem (0.1), (0.2), we obtain the following problem with parameter (ζ, τ) in the cone K:

$$\begin{gathered} L(D_y, \zeta, \tau)\widehat{u}(y, \zeta, \tau) = (\zeta^2 - \tau^2 - \Delta_y)\widehat{u}(y, \zeta, \tau) = \widehat{f}(y, \zeta, \tau), \\ \partial_\nu \widehat{u}(y, \zeta, \tau) = 0, \quad y \in \partial K, \end{gathered} \tag{2.1}$$

where ν is the unit outward normal to ∂K.

After the change of the variable $\eta = py$, the problem (2.1) takes the form

$$\begin{gathered} L(D_\eta, \theta)U(\eta, \zeta, \tau) = F(\eta, \zeta, \tau), \quad \eta \in K, \\ \partial_\nu U(\eta, \zeta, \tau) = 0, \quad \eta \in \partial K, \end{gathered} \tag{2.2}$$

where

$$\begin{gathered} U(\eta, \zeta, \tau) = \widehat{u}(p^{-1}\eta, \zeta, \tau), \quad F(\eta, \zeta, \tau) = p^{-2}\widehat{f}(p^{-1}\eta, \zeta, \tau), \\ \theta = \theta(\zeta, \tau) = (\zeta p^{-1}, \tau p^{-1}), \quad \tau = \sigma - i\gamma, \quad \sigma \in \mathbb{R}, \quad \gamma > 0. \end{gathered}$$

2.2. Energy estimate for solutions to the problem (2.1).

Proposition 2.1. *Let* $v \in S(\mathbb{R}^n)$, $\partial_\nu v(y) = 0$, $y \in \partial K$. *Then*

$$\gamma^2 \int_K (p^2 |v(y)|^2 + |\nabla_y v(y)|^2) \, dy \leqslant c \int_K |L(D_y, \zeta, \sigma - i\gamma) v(y)|^2 \, dy. \tag{2.3}$$

If $n - d > 2$, *then*

$$\gamma^2 \|v; H_0^1(K; p)\|^2 \leqslant c \|L(D_y, \zeta, \tau); L_2(K)\|^2. \tag{2.4}$$

The constant c *in* (2.3) *and* (2.4) *is independent of* $\zeta \in \mathbb{R}^d$, $\tau = \sigma - i\gamma$, $\sigma \in \mathbb{R}$, $\gamma > 0$.

Proof. In (1.8), we set

$$u(x,t) = \psi(z,t)v(y), \quad \psi \in \exp(\gamma t)S(\mathbb{R}^{d+1}_{(z,t)}) \cap S(\mathbb{R}^{d+1}_{(z,t)}),$$

where v satisfies the assumptions of the proposition. The inequality (1.8) takes the form

$$\gamma^2 \int |\widehat{\psi}(\zeta, \sigma - i\gamma)|^2 (p^2|v(y)|^2 + |\nabla_y v(y)|^2)\, dy\, d\zeta\, d\sigma$$
$$\leqslant c \int |\widehat{\psi}(\zeta, \sigma - i\gamma)|^2 |L(D_y, \zeta, \sigma - i\gamma) v(y)|^2 dy\, d\zeta\, d\sigma.$$

Since $\psi \in \exp(\gamma t)S(\mathbb{R}^{d+1}) \cap S(\mathbb{R}^{d+1})$ is arbitrary, we obtain (2.3).

In the case $n-d>2$, from the Hardy inequality

$$\int\limits_0^{+\infty} |w(r)|^2 r^{n-d-3}\, dr \leqslant \frac{4}{(n-d-2)^2} \int\limits_0^{+\infty} |\partial_r w(r)|^2 r^{n-d-1}\, dr$$

we obtain the inequality

$$\int\limits_K |v(y)|^2 |y|^{-2}\, dy \leqslant c \int\limits_K |\nabla v(y)|^2\, dy. \tag{2.5}$$

The inequality (2.4) is deduced from (2.3) and (2.5). □

Remark 2.2. The inequality (2.3) (as well as the inequality (2.4) in the case $n-d>2$) remains valid if for v we take a function of the form

$$v_k(y) = \chi(y)|y|^{i\lambda_k}\Phi_k(\omega),$$

where $\chi(y) = \chi^1(|y|)$, $\chi^1 \in C_c^\infty(\mathbb{R})$, $\chi^1 = 1$ in a neighborhood of O, $k = 0, 1, 2, \ldots$ for $n-d>2$ and $k = 1, 2, \ldots$ for $n-d=2$, λ_k and Φ_k were introduced in 1.1.

Indeed, analyzing the proof of the energy estimate (1.8) (cf. [2]), we see that the function

$$u(x,t) = \psi(z,t)v_k(y), \quad \psi \in \exp(\gamma t)S(\mathbb{R}^{d+1}) \cap S(\mathbb{R}^{d+1})$$

satisfies (1.8) if the function

$$(y,z,t) \mapsto \exp(-\gamma t)\Box u(y,z,t)$$

belongs to $L_2(Q)$. The last assertion is obviously true since $\Delta_y v_k = 0$ in a neighborhood of O and the functions v_k, as well as their first-order derivatives, are square summable ($\operatorname{Im}\lambda_k < (n-d-2)/2$).

2.3. Combined weight estimate for solutions to the problem (2.1) in the case $n-d>2$. We follow [2] and omit the proof. We denote by $\overset{0}{\mathfrak{D}}_\beta$ the linear set spanned by functions $w \in C_c^\infty(\overline{K} \setminus O)$ such that $\partial_\nu w(y) = 0$ for $y \in \partial K$ and functions v_k mentioned in Remark 2.2 such that $\operatorname{Im}\lambda_k < \beta - 1 + (n-d-2)/2$.

Proposition 2.3. *Suppose that $n-d>2$, $\beta \leqslant 1$, and the line* $\operatorname{Im}\lambda = \beta - 1 + (n-d-2)/2$ *does not contain points of the spectrum of the pencil $\mathfrak{A}(\lambda)$. Then for all $v \in \overset{0}{\mathfrak{D}}_\beta$ we have*

$$\|\chi_p v; H^2_\beta(K;p)\|^2 + \gamma^2\|v; H^1_\beta(K;p)\|^2 \leqslant c\{\|f; H^0_\beta(K)\|^2 + (p^{1-\beta}/\gamma)^2\|f; L_2(K)\|^2\}, \tag{2.6}$$

where $f = L(D_y, \zeta, \tau)v$, $\chi_p(y) = \chi(py)$, $\chi(y) = \chi^1(|y|)$, $\chi^1 \in C_c^\infty(\mathbb{R})$, $\chi^1 = 1$ *in a neighborhood of* O. *The constant* c *in* (2.6) *is independent of* $\zeta \in \mathbb{R}^d$, $\tau = \sigma - i\gamma$, $\sigma \in \mathbb{R}$, $\gamma > 0$.

The proof of Proposition 2.3 differs from the proof of the corresponding assertion in [2, Proposition 1.3] only by the fact that, instead of a local weight elliptic estimate for a solution to the Dirichlet problem for the operator $L(D_\eta, \theta)$, we use a similar estimate for a solution to the Neumann problem.

Proposition 2.4. *For any* $\beta \in \mathbb{R}$ *and* $U \in H_\beta^{s+1}(K;1)$ *such that* $\partial_\nu U = 0$ *on* ∂K *we have*

$$(\gamma/p)^2 \|\varkappa_\infty U; H_\beta^{s+1}(K;1)\|^2 \leqslant c\{\|\psi_\infty L(D_\eta, \theta)U; H_\beta^s(K;1)\|^2 + \|\psi_\infty U; H_{\beta-1}^{s+1}(K;1)\|^2\}, \tag{2.7}$$

where $\varkappa_\infty(y) = \varkappa_\infty^1(|y|)$, $\psi_\infty(y) = \psi_\infty^1(|y|)$, $\varkappa_\infty^1$, $\psi_\infty^1 \in C^\infty(\mathbb{R})$, $\varkappa_\infty^1 \psi_\infty^1 = \varkappa_\infty^1$, $\varkappa_\infty^1$ *and* ψ_∞^1 *vanish in a neighborhood of* O *and are equal to* 1 *at infinity,* $\theta = (\zeta/p, \tau/p)$, $s = 0, 1, \ldots$, *and the constant* c *is independent of the parameters.*

We make the following remark. In [2], the estimate (2.3) (but not the inequality (2.4) as was asserted there) was used in the proof of (2.7) in the case of the Dirichlet condition. This proof remains valid in the case of the Neumann condition; moreover, by the above arguments, (2.7) remains valid for $n - d = 2$.

Proposition 2.5 (cf. [19] or, for example, [20]). *Let* $\chi, \psi \in C_c^\infty(\overline{K})$, $\chi = 1$ *in a neighborhood of the vertex* O *of the cone* K, $\chi\psi = \chi$. *Let the line* $\operatorname{Im}\lambda = \beta - s - 1 + (n - d - 2)/2$ *do not contain eigenvalues of the pencil* $\mathfrak{A}$. *Then every function* $U \in H_\beta^{s+2}(K;1)$ *such that* $\partial_\nu U = 0$ *on* ∂K *satisfies the inequality*

$$\|\chi U; H_\beta^{s+2}(K;1)\|^2 \leqslant c\{\|\psi L(D_\eta, \theta)U; H_\beta^s(K;1)\|^2 + \|\psi U; H_\beta^{s+1}(K;1)\|^2\}. \tag{2.8}$$

Proceeding by induction on q and using Proposition 2.3 at the first step (cf. [2]), from (2.7) and (2.8) we obtain the following assertion.

Proposition 2.6. *We set*

$$\{f\}_{q,\beta}(K;p,\gamma)^2 = \sum_{j=0}^{q} (p/\gamma)^{2j} \|f; H_{\beta+q-j}^{q-j}(K;p)\|^2 + (p^{1-\beta+q}/\gamma^{1+q})^2 \|f; L_2(K)\|^2, \tag{2.9}$$

where $\beta \in \mathbb{R}$ *and* $q = 0, 1, \ldots$. *Let* $\beta \leqslant 1$, *and let the line* $\operatorname{Im}\lambda = \beta - 1 + (n - d - 2)/2$ *do not contain points of the spectrum of the pencil* $\overset{0}{\mathfrak{A}}$. *Then* $v \in \mathfrak{D}_\beta$ *satisfies the inequality*

$$\|\chi_p v; H_{\beta+q}^{2+q}(K;p)\|^2 + \gamma^2 \|v; H_{\beta+q}^{1+q}(K;p)\|^2 \leqslant c\{L(D_y, \zeta, \tau)v\}_{q,p}(K;p,\gamma)^2, \tag{2.10}$$

where $q = 0, 1, \ldots$ *and the constant* c *is independent of* v *and the parameters.*

§ 3. Combined Estimate for a Plane Corner

In this section, K is a corner of angle α in $\mathbb{R}_x^2$ and $r = |x|$. We introduce the space $H_\beta^{l,0}(K)$ equipped with the norm

$$\|u; H_\beta^{l,0}(K)\| = \|r^{\beta-l+1}u; L_2(K)\| + \sum_{0<|\alpha|\leqslant l} \|r^{\beta-l+|\alpha|} D^\alpha u; L_2(K)\|. \tag{3.1}$$

Similar spaces with nonhomogeneous norms were introduced in [21, 5, 22]. The same spaces were introduced in [20, Sec. 5.5] (cf. also [4]). For the sake of convenience, we prove all necessary technical facts because the required result (the estimate (3.7) below) was not formulated in [20, 4] explicitly. In fact, we follow [21] (for details we refer the reader to [22]).

Lemma 3.1. *Let q be a nonnegative integer, and let $u \in H^{2+q,0}_{\beta+q}$, $\beta < 1$. The following assertions hold.*

1. $\chi u = \chi(v+A)$, *where* $\chi \in C_c^\infty(\mathbb{R}^2)$, $\chi = 1$ *in a neighborhood of* O, $v \in H^{2+q}_{\beta+q}(K)$, $A \in \mathbb{C}$.

2. *The following estimate holds*:

$$|A| \leqslant c\{\|r^{\beta-1}\nabla u; L_2(K\cap\{|x|\leqslant 1\})\| + \|u; L_2(K\cap\{1/2\leqslant |x|\leqslant 1\})\|\}. \tag{3.2}$$

3. *If* $\beta \leqslant 0$, *then* $A = 0$.

Proof. 1. Let

$$A(t) = \alpha^{-1}\int_0^\alpha u(t\cos\omega, t\sin\omega)\,d\omega, \quad A = A(0+).$$

The existence of the last limit follows from the estimate

$$\begin{aligned}|A(t_2)-A(t_1)| &\leqslant \int_{t_1}^{t_2}|A'(t)|\,dt \leqslant \left(\int_{t_1}^{t_2}|A'(t)|^2 t^{2\beta-1}\,dt\right)^{1/2}\left(\int_0^T t^{1-2\beta}\,dt\right)^{1/2}\\ &\leqslant c(T)\left(\int_{t_1\leqslant|x|\leqslant t_2}|\nabla u|^2 r^{2\beta-2}\,dx\right)^{1/2}, \quad 0<t_1,t_2\leqslant T,\end{aligned}$$

and the inclusion

$$r^{\beta-1}\nabla u \in L_2(K).$$

Assertion 1 follows from the estimate

$$\|r^{\beta-2}(u-A); L_2(K\cap\{|x|\leqslant 1\})\| \leqslant c\|r^{\beta-1}\nabla u; L_2(K\cap\{|x|\leqslant 1\})\|. \tag{3.3}$$

Let us prove (3.3). We have

$$\int_0^1 dr\, r^{2\beta-3}\int_0^\alpha |u-A|^2\,d\omega \leqslant c\left\{\int_0^1 dr\, r^{2\beta-3}\int_0^\alpha |u-A(r)|^2\,d\omega + \int_0^1 r^{2\beta-3}|A(r)-A(0)|^2\,dr\right\}. \tag{3.4}$$

We estimate the second term on the right-hand side of (3.4):

$$\int_0^1 r^{2\beta-3}|A(r)-A(0)|^2\,dr \leqslant 2\int_0^1 dr\, r^{2\beta-3}\int_0^r |A'(\tau)|\,|A(\tau)-A(0)|\,d\tau$$

$$= 2\int_0^1 d\tau |A'(\tau)|\,|A(\tau)-A(0)|\int_\tau^1 r^{2\beta-3}\,dr$$

$$\leqslant c\int_0^1 |A'(\tau)|\,|A(\tau)-A(0)|\tau^{2\beta-2}\,d\tau$$

$$\leqslant c\Big(\int_0^1 |A(\tau)-A(0)|^2\tau^{2\beta-3}\,d\tau\Big)^{1/2}\Big(\int_0^1 \tau^{2\beta-1}|A'(\tau)|^2\,d\tau\Big)^{1/2}.$$

Hence

$$\int_0^1 r^{2\beta-3}|A(r)-A(0)|^2\,dr \leqslant c\int_0^1 r^{2\beta-1}|A'(r)|^2\,dr. \tag{3.5}$$

By the definition of $A(t)$, we find

$$\int_0^1 r^{2\beta-3}|A(r)-A(0)|^2\,dr \leqslant c\|r^{\beta-1}\nabla u; L_2(K\cap\{|x|\leqslant 1\})\|^2.$$

The same estimate for the first term on the right-hand side of (3.4) follows from the Poincaré inequality on the arc $[0,\alpha]$.

2. We have

$$|A(0)| \leqslant |A(t)| + \int_0^t |A'(\tau)|\,d\tau \leqslant |A(t)| + \Big(\int_0^1 |A'(\tau)|^2\tau^{2\beta-1}\,d\tau\Big)^{1/2}\Big(\int_0^1 \tau^{1-2\beta}\,d\tau\Big)^{1/2}.$$

Hence

$$|A(0)| \leqslant c\{|A(t)| + \|r^{\beta-1}\nabla u; L_2(K\cap\{|x|\leqslant 1\})\|\}.$$

Integrating the last inequality with respect to $t\in[1/2,1]$, we obtain (3.2).

3. From (3.1) and (3.3) it follows that the functions $r^{\beta-1}u$ and $r^{\beta-2}(u-A)$ belong to the class $L_2(K\cap\{|x|\leqslant 1\})$. Consequently,

$$Ar^{\beta-1}\in L_2(K\cap\{|x|\leqslant 1\}).$$

In the case $\beta\leqslant 0$, the last assertion is valid only if $A=0$. □

Lemma 3.2. *Under the assumptions of Lemma* 3.1, *for any* $\varepsilon\in(0,1)$ *the following estimate holds*:

$$|A| \leqslant \varepsilon\|r^{\beta-1}\nabla u; L_2(K\cap\{|x|\leqslant 1\})\| + C_\varepsilon\|u; L_2(K\cap\{\varepsilon^{1/(1-\beta)}/2\leqslant |x|\leqslant 1\})\|. \tag{3.6}$$

Proof. We apply the estimate (3.2) to the function $w(x) = u(\varepsilon x)$. Then

$$|A|^2 \leqslant c\left\{\varepsilon^2 \int\limits_{K\cap\{|x|\leqslant 1\}} |x|^{2\beta-2}|\nabla u(\varepsilon x)|^2\,dx + \int\limits_{K\cap\{1/2\leqslant|x|\leqslant 1\}} |u(\varepsilon x)|^2\,dx\right\}$$
$$\leqslant c\left\{\varepsilon^{2-2\beta} \int\limits_{K\cap\{|x|\leqslant 1\}} |x|^{2\beta-1}|\nabla u(x)|^2\,dx + \varepsilon^{-2} \int\limits_{K\cap\{\varepsilon/2\leqslant|x|\leqslant 1\}} |u(x)|^2\,dx\right\}.$$

Replacing ε with $\varepsilon^{1/(1-\beta)}$, we obtain (3.6). □

Proposition 3.3. *Let $\beta < 1$, and let the line $\operatorname{Im}\lambda = \beta - 1$ do not contain points of the spectrum of the pencil $\mathfrak{A}$. Let $\chi \in C_c^\infty(\mathbb{R}^2)$, $\chi(x) = 1$ for $|x| \leqslant 1$, $\chi(x) = 0$ for $|x| \geqslant 3/2$. Let q be a nonnegative integer. Then for some $\delta > 0$ and any function $u \in H_{\beta+q}^{2+q,0}(K)$ satisfying the homogeneous Neumann condition on ∂K the following estimate hold:*

$$\|\chi u; H_{\beta+q}^{q+2,0}(K)\| \leqslant c\{\|\chi L(D_\eta,\theta)u; H_{\beta+q}^{q}(K)\| + \|u; H^{q+1}(K\cap\{\delta \leqslant |x| \leqslant 2\}\|\}. \tag{3.7}$$

Proof. As above, $\chi u = \chi v + \chi A$, where the function χv belongs to $H_{\beta+q}^{2+q}(K)$ and satisfies the homogeneous Neumann condition on ∂K (we can assume that the cut-off function χ depends only on $|x|$). For all β such that the line $\operatorname{Im}\lambda = \beta - 1$ does not contain points of the spectrum of the pencil $\mathfrak{A}$ the following estimate in the homogeneous norms holds:

$$\|\chi v; H_{\beta+q}^{q+2}(K)\| \leqslant c\{\|\chi L(D_\eta,\theta)v; H_{\beta+q}^{q}(K)\| + \|\psi v; H_{\beta+q}^{q+1}(K)\|\}, \tag{3.8}$$

where $\psi \in C_c^\infty(\mathbb{R}^2)$, $\psi\chi = \chi$ (cf. [19] or [20]). By (3.8), we have

$$\|\chi u; H_{\beta+q}^{q+2,0}(K)\| \leqslant c(\|\chi v; H_{\beta+q}^{q+2}(K) + \|\chi A; H_{\beta+q}^{q+2,0}(K)\|)$$
$$\leqslant c\{\|\chi L(D_\eta,\theta)v; H_{\beta+q}^{q}(K)\| + \|\psi v; H_{\beta+q}^{q+1}(K)\| + |A|\} \tag{3.9}$$

(recall that $A = 0$ if $\beta \leqslant 0$). Since

$$\|\chi L(D_\eta,\theta)v; H_{\beta+q}^{q}(K)\| \leqslant \|\chi L(D_\eta,\theta)u; H_{\beta+q}^{q}(K)\| + \|\chi L(D_\eta,\theta)A; H_{\beta+q}^{q}(K)\|$$
$$\leqslant c(\|\chi L(D_\eta,\theta)u; H_{\beta+q}^{q}(K)\| + |A|),$$

we have

$$\|\chi u; H_{\beta+q}^{q+2,0}(K)\| \leqslant c(\|\chi L(D_\eta,\theta)u; H_{\beta+q}^{q}(K)\| + \|\psi v; H_{\beta+q}^{q+1}(K)\| + |A|). \tag{3.10}$$

We estimate the second term on the right-hand side of (3.10). Using (3.3), we find

$$\begin{aligned}
\|\psi v; H^{q+1}_{\beta+q}(K)\|^2 &\leqslant c \int\limits_{K\cap\{|x|\leqslant 2\}} \{|v|^2 r^{2\beta-2} + |\nabla u|^2 r^{2\beta} + \dots + |\nabla^{q+1} u|^2 r^{2\beta+2q}\}\, dx \\
&\leqslant c\{\varepsilon^2 \int\limits_{K\cap\{|x|\leqslant \varepsilon\}} (|u-A|^2 r^{2\beta-4} + |\nabla u|^2 r^{2\beta-2} + \dots + |\nabla^{q+1} u|^2 r^{2\beta+2q-2})\, dx \\
&\quad + \int\limits_{K\cap\{\varepsilon\leqslant|x|\leqslant 2\}} (|u-A|^2 r^{2\beta-2} + |\nabla u|^2 r^{2\beta} + \dots + |\nabla^{q+1} u|^2 r^{2\beta+2q})\, dx\} \\
&\leqslant c\Big\{\varepsilon^2 \int\limits_{K\cap\{|x|\leqslant 1\}} (|\nabla u|^2 r^{2\beta-2} + \dots + |\nabla^{q+1} u|^2 r^{2\beta+2q-2})\, dx + |A| \\
&\quad + C_\varepsilon \|u; H^{q+1}(K\cap\{\varepsilon \leqslant |x| \leqslant 2\})\|\Big\} \\
&\leqslant c\{\varepsilon^2 \|\chi u; H^{q+2,0}_{\beta+q}(K)\| + |A| + C_\varepsilon \|u; H^{q+1}(K\cap\{\varepsilon\leqslant |x| \leqslant 2\})\|\}.
\end{aligned}$$

Estimating $|A|$ with the help of Lemma 3.2, taking a sufficiently small ε, and moving the terms containing ε to the left-hand side of (3.10), we obtain the inequality (3.7). □

Let $\chi \in C_c^\infty(\mathbb{R}^2)$, $\chi = 1$ in a neighborhood of O, and let $\beta < 1$. We denote by $\overset{0}{\mathfrak{D}}_\beta(2)$ the lineal spanned on functions $u \in C_c^\infty(\overline{K}\setminus O)$ such that $\partial_\nu u = 0$ on ∂K, on functions $\chi(x)|x|^{i\lambda_j}\Phi_j(x/|x|)$, where $\operatorname{Im}\lambda_j < \beta - 1$, and (only if $\beta \in (0,1)$) on the function $\chi\Phi_0 = \chi\alpha^{-1/2}$. If $\beta \leqslant 0$, then the lineal $\overset{0}{\mathfrak{D}}_\beta(2)$ does not contain the function $\chi\Phi_0$.

Proposition 3.4. *Let* $\beta < 1$, *let the line* $\operatorname{Im}\lambda = \beta - 1$ *do not contain points of the spectrum of the pencil* $\mathfrak{A}$, *and let* q *be a nonnegative integer. Then every function* u *of the lineal* $\overset{0}{\mathfrak{D}}_\beta(2)$ *satisfies the following estimate*:

$$\begin{aligned}
&\|\chi u; H^{q+2,0}_{\beta+q}(K)\|^2 + (\gamma/p)^2 \|u; H^{q+1}_{\beta+q}(K;1)\|^2 \\
&\qquad\leqslant c\{\|L(D_\eta,\theta)u; H^q_{\beta+q}(K;1)\|^2 + \|\psi_\infty u; H^{q+1}_{\beta+q-1}(K;1))\|^2\}, \quad (3.11)
\end{aligned}$$

where $\psi_\infty \in C^\infty(\mathbb{R}^2)$, $\psi_\infty = 0$ *in a neighborhood of* O, *and* $\psi_\infty = 1$ *at infinity.*

Proof. Adding (2.7) (for $s = q$ and $\beta = \beta + q$) and (3.7), we obtain the inequality

$$\begin{aligned}
\|\chi u; H^{q+2,0}_{\beta+q}(K)\|^2 + (\gamma/p)^2 \|\varkappa_\infty u; H^{q+1}_{\beta+q}(K;1)\|^2 &\leqslant c\{\|L(D_\eta,\theta)u; H^q_{\beta+q}(K;1)\|^2 \\
&\quad + \|u; H^{q+1}(K\cap\{\delta \leqslant |x| \leqslant 2\}\|^2 + \|\psi_\infty u; H^{q+1}_{\beta+q-1}(K;1)\|^2\}. \quad (3.12)
\end{aligned}$$

Since $\gamma/p < 1$, we have

$$(\gamma/p)^2 \|(1-\varkappa_\infty)u; H^{q+1}_{\beta+q}(K;1)\|^2 \leqslant c\|\chi u; H^{2+q,0}_{\beta+q}(K)\|^2.$$

Hence the cut-off function $\varkappa_\infty$ on the left-hand side of (3.12) is not required. If the support of the function $1-\psi_\infty$ is sufficiently small, then the second term on the right-hand side of (3.12) is estimated from above by $c\|\psi_\infty u; H^{q+1}_{\beta+q-1}(K;1)\|^2$. □

Making the change of variables $\eta \to y = \eta p^{-1}$ in (3.11), we obtain the estimate

$$p^2\|r^{\beta-1}\chi_p v; L_2(K)\|^2 + \sum_{s=1}^{q+2} \|r^{\beta-2+s}\nabla^s(\chi_p v); L_2(K)\|^2 + \gamma^2\|v; H^{q+1}_{\beta+q}(K;p)\|^2$$
$$\leqslant c\{\|L(D_y,\zeta,\tau)v; H^q_{\beta+q}(K;p)\|^2 + \|\psi_{\infty,p}v; H^{q+1}_{\beta+q-1}(K;p)\|^2\}, \quad (3.13)$$

where

$$\chi_p(y) = \chi(py), \quad \psi_{\infty,p}(y) = \psi_\infty(py), \quad v(y) = u(py).$$

We estimate the last term on the right-hand side of (3.13) for $q = 0$. We have

$$\|\psi_{\infty,p}v; H^1_{\beta-1}(K;p)\|^2 \leqslant c \int\limits_{c_1/p<|y|} |y|^{2(\beta-1)}(|\nabla v|^2 + |y|^{-2}|v|^2 + p^2|v|^2)\,dy$$
$$\leqslant cp^{2(1-\beta)}\int\limits_K (|\nabla v|^2 + p^2|v|^2)\,dy \leqslant c(p^{1-\beta}/\gamma)^2\|L(D_y,\zeta,\tau)v; L_2(K)\|^2. \quad (3.14)$$

(At the last step, we used the estimate (2.3).)

Now, we are ready to prove the main a priori estimate.

Proposition 3.5. *Let $\beta < 1$, let the line $\operatorname{Im}\lambda = \beta - 1$ do not contain any point of the spectrum of the pencil $\mathfrak{A}$, and let q be a nonnegative integer. Then every function $u \in \overset{0}{\mathfrak{D}}_\beta(2)$ satisfies the estimate*

$$p^2\|r^{\beta-1}\chi_p v; L_2(K)\|^2 + \sum_{k=1}^{q+2} \|r^{\beta-2+k}\nabla^k(\chi_p v); L_2(K)\|^2 + \gamma^2\|v; H^{q+1}_{\beta+q}(K;p)\|^2$$
$$\leqslant c\left\{\sum_{j=0}^{q}(p/\gamma)^{2j}\|L(D_y,\zeta,\tau)v; H^{q-j}_{\beta+q-j}(K;p)\|^2 + (p^{1-\beta+q}/\gamma^{q+1})^2\|L(D_y,\zeta,\tau)v; L_2(K)\|^2\right\}. \quad (3.15)$$

Proof. We denote by $\{Lv\}^2_q$ the right-hand side of (3.15). For $q = 0$ the inequality (3.15) follows from (3.13) and (3.14). Assuming that the inequality (3.15) holds for $q \leqslant q_0$, we prove that (3.15) holds for $q = q_0 + 1$. By Proposition 3.4, it is necessary to estimate the norm

$$\|\psi_{\infty,p}v; H^{q_0+2}_{\beta+q_0}(K;p)\|^2 = \|\psi_{\infty,p}v; H^{q_0+2}_{\beta+q_0}(K)\|^2 + p^2\|\psi_{\infty,p}v; H^{q_0+1}_{\beta+q_0}(K;p)\|^2. \quad (3.16)$$

To estimate the first term on the right-hand side of (3.16), we write the equality

$$L(D_y,0,0)(\psi_{\infty,p}v) = \psi_{\infty,p}L(D_y,\zeta,\tau)v + [L(D_y,\zeta,\tau),\psi_{\infty,p}]v + (L(D_y,0,0) - L(D_y,\zeta,\tau))\psi_{\infty,p}v. \quad (3.17)$$

Since the line $\operatorname{Im}\lambda = \beta - 1$ contains no eigenvalues of the pencil $\mathfrak{A}$, the equation

$$L(D_y,0,0)w = f \in H^{q_0}_{\beta+q_0}(K)$$

has a unique solution $w \in H^{q_0+2}_{\beta+q_0}(K)$ satisfying the homogeneous Neumann condition on ∂K. The following estimate holds:

$$\|w; H^{q_0+2}_{\beta+q_0}(K)\| \leqslant c\|f; H^{q_0}_{\beta+q_0}(K)\| \quad (3.18)$$

(cf. [19] or [20]). The norms of the second and third terms on the right-hand side of (3.17) do not exceed $cp\|v;H^{q_0+1}_{\beta+q_0}(K;p)\|$. Therefore, from (3.16), (3.17), and (3.18) it follows that

$$\|\psi_{\infty,p}v;H^{q_0+2}_{\beta+q_0}(K;p)\|^2 \leqslant c(\ \|L(D_y,\zeta,\tau)v;H^{q_0}_{\beta+q_0}(K)\| + p^2\|v;H^{q_0+1}_{\beta+q_0}(K;p)\|^2).$$

The last summand was already estimated at the previous step of the induction process (cf. (3.15) for $q=q_0$):

$$p^2\|v;H^{q_0+1}_{\beta+q_0}(K;p)\|^2 \leqslant (p/\gamma)^2\{Lv\}^2_{q_0}.$$

It remains to note that

$$(p/\gamma)^2\{Lv\}^2_{q_0} + \|Lv;H^{q_0+1}_{\beta+q_0+1}(K;p)\|^2 = \{Lv\}^2_{q_0+1}.$$

§ 4. The Operator of the Boundary-Value Problem in a Cone

4.1. A weak solution to the problem (2.1). We write the problem (2.1) in the simplified notation:

$$\begin{gathered} L(D_y,\zeta,\tau)u = (\zeta^2-\tau^2-\Delta_y)u = f \quad \text{in } K, \\ \partial_\nu u = 0 \quad \text{on } \partial K. \end{gathered} \tag{4.1}$$

As usual, a function $u\in H^1(K)$ is called a *weak solution to the problem* (4.1) with $f\in L_2(K)$ if the following integral identity holds:

$$B(u,v) := \int\limits_K u_y\bar{v}_y - (\tau^2-\zeta^2)u\bar{v}\,dy = \int\limits_K f\bar{v}\,dy$$

for any function $v\in H^1(K)$. By the Vishik–Lax–Milgram theorem[1] and the following assertion, we obtain the existence of a weak solution to the problem (4.1) for all $f\in L_2(K)$, $\zeta\in\mathbb{R}^d$, $\tau=\sigma-i\gamma$, $\gamma\neq 0$.

Proposition 4.1. *For* $\tau^2-\zeta^2\in\mathbb{C}\setminus\overline{\mathbb{R}}_+$ *we have*

$$|B(u,u)| \geqslant \delta\|u;H^1(K)\|^2, \tag{4.2}$$

where $\delta=\delta(\tau,\zeta)>0$.

Proof. Let

$$\alpha = \|u;L_2(K)\|^2, \quad \beta = \|u_y;L_2(K)\|^2, \quad \tau^2-\zeta^2 = \sigma_1+i\gamma_1.$$

Then

$$|B(u,u)|^2 = \beta^2 + (\sigma_1^2+\gamma_1^2)\alpha^2 - 2\sigma_1\alpha\beta.$$

If $\sigma_1=0$, $\gamma_1\neq 0$ or $\sigma_1<0$, then the inequality (4.2) is obvious. Let $\sigma_1>0$. Then $\gamma_1\neq 0$. Let ε be such that $\varepsilon^2<1$ and $(1/\varepsilon^2-1)\sigma_1^2<\gamma_1^2$. Then

$$2\sigma_1\alpha\beta \leqslant \varepsilon^2\beta^2 + (1/\varepsilon^2)\sigma_1^2\alpha^2,$$

[1] See the version of this result in [23, Remark 2.9.3].

$$|B(u,u)|^2 \geqslant \beta^2(1-\varepsilon^2)+\alpha^2[\gamma_1^2-(1/\varepsilon^2-1)\sigma_1^2] \geqslant \delta(\alpha^2+\beta^2).$$

The proposition is proved. □

4.2. A strong solution to the problem (2.1). We introduce an unbounded operator $A(\zeta,\tau)$ in $L_2(K)$ with domain

$$D(A)=\begin{cases} \overset{0}{\mathcal{D}}_1, & n-d>2, \\ \overset{0}{\mathcal{D}}_\beta(2), & n-d=2, \end{cases}$$

where β is taken from the interval $(\max\{0,\operatorname{Im}\lambda_1+1\},1)$, where $\lambda_1=-(\pi/\alpha)i$ and α is the opening angle of the corner K. We note that $D(A)$ is independent of the choice of β. The operator $A(\zeta,\tau)$ is defined by the formula

$$D(A) \ni v \mapsto A(\zeta,\tau)v = L(D_y,\zeta,\tau).$$

This operator admits the closure which will be denoted in the same way. In the sequel, we use only the closed operator $A(\zeta,\tau)$. The domain of this operator is denoted by $D(A(\zeta,\tau))$. Proposition 2.1 and Remark 2.2 lead to the following assertion.

Proposition 4.2. 1. *If $n-d>2$, then $D(A(\zeta,\tau)) \subset H_0^1(K;p)$. In the case $n-d=2$, we have*

$$D(A(\zeta,\tau)) \subset H^1(K).$$

2. $\operatorname{Ker} A(\zeta,\tau)=0$.

3. *The lineal* $\operatorname{Im} A(\zeta,\tau)$ *is closed in* $L_2(K)$.

Our next goal is to show that $\operatorname{Im} A(\zeta,\tau)=L_2(K)$. We need some results of the theory of elliptic problems in domains with conical points (cf. [19] or, for example, [20]).

We consider the homogeneous problem

$$\begin{aligned} v&=0 \quad \text{in } K, \\ \partial_\nu v&=0 \quad \text{on } \partial K. \end{aligned} \tag{4.3}$$

Let $r=|y|$, and let $\omega=y/|y| \in S^{n-d-1}$. For every eigenvalue $\lambda_{\pm k}$ of the pencil $\mathfrak{A}$ we can construct a special solution $w_{\pm k}$ to the problem (4.3) that has the following asymptotic behavior:

$$w_{\pm k} \sim r^{i\lambda_{\pm k}}\Psi(\omega)$$

in a neighborhood of O. For $n-d=2$, starting with the double eigenvalue 0 of the pencil $\mathfrak{A}$, we construct two special solutions w_0 and w_{01} to the problem (4.3) with asymptotics $\alpha^{-1/2}$ and $\alpha^{-1/2}\ln r$ (recall that α is the opening angle of the corner K in the plane $\mathbb{R}^2$). Namely (cf. [6]),

$$w_{-k}(y)=\frac{2^{1-\nu_k}}{\Gamma(\nu_k)}(i|y|\sqrt{-|\zeta|^2+\tau^2})^{\nu_k}K_{\nu_k}(i|y|\sqrt{-|\zeta|^2+\tau^2})|y|^{i\lambda_{-k}}\Phi_k(y/|y|), \tag{4.4}$$

$$w_{+k}(y)=\Gamma(1+\nu_k)2^{\nu_k}(i|y|\sqrt{-|\zeta|^2+\tau^2})^{-\nu_k}I_{\nu_k}(i|y|\sqrt{-|\zeta|^2+\tau^2})|y|^{i\lambda_{+k}}\Phi_k(y/|y|). \tag{4.5}$$

Here, $\nu_k = \sqrt{(n-d-2)^2+4\mu_k}/2$, K_ν and I_ν are the modified Bessel functions of the third and first kind respectively,

$$\sqrt{z} = \exp((1/2)\ln|z| + i\arg z/2 + i\pi), \quad \arg z \in [0,2\pi).$$

For $n-d=2$ we have

$$w_0(y) = \alpha^{-1/2} I_0(i|y|\sqrt{-|\zeta|^2+\tau^2}), \tag{4.6}$$

$$w_{01}(y) = \alpha^{-1/2} K_0(i|y|\sqrt{-|\zeta|^2+\tau^2}). \tag{4.7}$$

Lemma 4.3 (cf. [19] or [20]). *Let $\chi \in C_c^\infty(\mathbb{R}^{n-d})$, $\chi = 1$ in a neighborhood of O. Let a function u be such that $\chi u \in H^2_\beta(K)$ for some $\beta \in \mathbb{R}$, $L(D_y,\zeta,\tau)u = f$ in K, and $\partial_\nu u = 0$ on ∂K, where $f \in C_c^\infty(K)$. Then we have the following asymptotic expansion:*

$$\chi u \sim \sum_{\pm k} c_{\pm k}\chi w_{\pm k}, \tag{4.8}$$

where the summation on the right-hand side of (4.8) *is taken over $\pm k$ such that $\chi w_{\pm k} \in H^2_\beta(K)$. (For $n-d=2$ the sum can contain the term $c_{01}\chi w_{01}$.) To obtain χu with accuracy of order up to $O(r^\Lambda)$ in a neighborhood of the origin, it is necessary to take only those terms on the right-hand side of* (4.8) *that decrease in a neighborhood of O slower than $O(r^\Lambda)$.*

Proposition 4.4.

$$\operatorname{Im} A(\zeta,\tau) = L_2(K).$$

Proof. By Assertion 3 of Proposition 4.2, it suffices to show that $C_c^\infty(K) \subset \operatorname{Im} A(\zeta,\tau)$.

Let $f \in C_c^\infty(K)$, and let u be a weak solution to the problem (4.1). By the known results of the theory of elliptic boundary-value problems (cf., for example, [24]), we have $u \in C^\infty(\overline{K}\setminus O)$, $L(D_y,\zeta,\tau)u = 0$ in K, and $\partial_\nu u = 0$ on $\partial K \setminus O$. To prove the proposition, it suffices to check the inclusion $u \in D(A(\zeta,\tau))$. □

Lemma 4.5.

$$\int_{\{y\in K:\ |y|<1\}} (|y|^2|\nabla u(y)|^2 + |y|^4|\nabla^2 u(y)|^2)\,dy < +\infty, \tag{4.9}$$

$$\int_{\{y\in K:\ |y|>1\}} (|\nabla u(y)|^2/|y|^2 + |\nabla^2 u(y)|^2)\,dy < +\infty. \tag{4.10}$$

Proof (cf. [1, Lemma 3.4]). Let $\psi, \varkappa \in C^\infty(\overline{K})$. We assume that $\psi\varkappa = \varkappa$, $\operatorname{supp}\varkappa \subset \{y\colon 1/4 < |y| < 4\}$. Any function $u \in H^2_{\mathrm{loc}}(\overline{K}\setminus O)$ such that $\partial_\nu u = 0$ on ∂K satisfies the following estimate (cf., for example, [24] or [25]):

$$\|\varkappa u; H^2(K)\| \leqslant c\{\|\psi\Delta u; L_2(K)\| + \|\psi u; L_2(K)\|\}. \tag{4.11}$$

We introduce the partition of unity $\{\varkappa_j\}_{j=-\infty}^{j=+\infty}$ and functions $\psi_j \in C^\infty(\overline{K}\setminus O)$ such that

$$\operatorname{supp}\varkappa_j \subset \{y:\ 2^{j-1} < |y| < 2^{j+1}\},$$
$$\operatorname{supp}\psi_j \subset \{y:\ 2^{j-2} < |y| < 2^{j+2}\},$$

$\varkappa_j\psi_j = \varkappa_j$; moreover,

$$|D^\alpha \varkappa_j| + |D^\alpha \psi_j| \leqslant C_\alpha 2^{-j|\alpha|}.$$

Making the change of variables $y \mapsto Y = 2^j y$, from the estimate (4.11) we deduce the inequality

$$\sum_{|\alpha|\leqslant 2} 2^{j(2|\alpha|-4)} \int_K |D_y^\alpha(\varkappa_j u)|^2\,dy \leqslant c\Big\{\int_K |\psi_j \Delta u|^2\,dy + 2^{-4j}\int_K |\psi_j u|^2\,dy\Big\}. \tag{4.12}$$

Taking into account the equality $\Delta u = (|\zeta|^2 - \tau^2)u - f$ and adding the inequalities (4.12) for $j = 1, 2, \ldots$, we arrive at the estimate (4.10).

To obtain (4.9), we multiply (4.12) by 2^{4j} and take the sum over $j = 0, -1, \ldots$. □

It is easy to check that $u \in D(A(\zeta,\tau))$. Let χ be the function from Lemma 4.3. We set $\psi = 1 - \chi$ and $u = \chi u + \psi u$. From the inclusion $u \in H^1(K)$ and Lemmas 4.5 and 4.3 it follows that

$$\chi u = \chi \sum c_k r^{i\lambda_k}\Phi_k(\omega) + v, \tag{4.13}$$

where the sum contains only those terms for which $(n-d-2)/2 - 1 < \operatorname{Im}\lambda_k < (n-d-2)/2$ and $v = O(r^{1-(n-d-2)/2+\varepsilon})$ for some $\varepsilon > 0$. For $n - d = 2$ the sum can contain the term $c_0\Phi_0(\omega) \equiv c_0\alpha^{-1/2}$. The functions $\chi r^{i\lambda_k}\Phi_k(\omega)$ from the sum in (4.13) belong to $D(A(\zeta,\tau))$ by definition. We show that $v \in D(A(\zeta,\tau))$. Let $\psi_n(x) = \psi(nx)$. We have

$$L(D_y,\zeta,\tau)(\psi_n v) = \psi_n L(D_y,\zeta,\tau)v + 2\nabla\psi_n\nabla v + \Delta\psi_n v. \tag{4.14}$$

It is clear that

$$L(D_y,\zeta,\tau)v = L(D_y,\zeta,\tau)\{\chi u - \chi\sum c_k r^{i\lambda_k}\Phi_k(\omega)\} \in L_2(K).$$

Therefore, the first term on the right-hand side of (4.14) tends to $L(D_y,\zeta,\tau)v$ in $L_2(K)$ as $n \to \infty$, whereas the second and third terms tend to zero in $L_2(K)$ because

$$\|\nabla\psi_n\nabla v\|^2 \leqslant cn^2 \int_{c_1/n}^{c_2/n} r^{n-d-1} r^{2\varepsilon-(n-d-2)}\,dr,$$

$$\|\Delta\psi_n v\|^2 \leqslant cn^4 \int_{c_1/n}^{c_2/n} r^{n-d-1} r^{2-(n-d-2)+2\varepsilon}\,dr.$$

Thus, $\psi_n v \to v$ and $L(D_y,\zeta,\tau)(\psi_n v) \to L(D_y,\zeta,\tau)v$. Therefore, $v \in D(A(\zeta,\tau))$. Consequently, $\chi u \in D(A(\zeta,\tau))$. The inclusion $\psi u \in D(A(\zeta,\tau))$ is established with the help of the sequence $\chi_n\psi u$, where $\chi_n \in C_c^\infty(\mathbb{R}^{n-d})$, $\chi_n(x) = 1$ for $|x| \leqslant n$, and $\chi_n(x) = 0$ for $|x| \geqslant n+1$. □

Definition 4.6. By a *strong solution* to the problem (4.1) with $f \in L_2(K)$ we mean a solution u to the equation $A(\zeta,\tau)u = f$.

Propositions 4.2 and 4.4 imply the following assertion.

Theorem 4.7. *For any $f \in L_2(K)$, $\gamma > 0$, $\zeta \in \mathbb{R}^d$, and $\sigma \in \mathbb{R}$ there exists a unique strong solution u to the problem* (4.1). *For $n - d > 2$ we have*

$$\gamma^2 \| v; H_0^1(K;p)\|^2 \leqslant c\|A(\zeta,\tau)v; L_2(K)\|^2, \tag{4.15}$$

and for $n - d = 2$ we have

$$\gamma^2 \int_K (p^2|v(y)|^2 + |\nabla_y v(y)|^2)\, dy \leqslant c\|A(\zeta,\tau)v; L_2(K)\|^2. \tag{4.16}$$

In (4.15) *and* (4.16), *the constant c is independent of γ, ζ and σ.*

4.3. Formulas for the coefficients in the asymptotic expansion (4.8). Let $f \in C_c^\infty(K)$, and let u be a strong solution to the problem (4.1). Then

$$\chi u \sim \sum_k c_k \chi w_{+k},$$

where only the summands with nonnegative k occur in the sum. The following assertion follows from the general results [26] (cf., also [20]) about the coefficients of asymptotic expansions of solutions to elliptic problems in a cone.

Lemma 4.8. *We denote by $w_k(,\overline{\tau})$ and $w_{01}(,\overline{\tau})$ the functions from* (4.4) *and* (4.7) *with $\overline{\tau}$ instead of τ. Then*

$$c_k = (f, w_k(,\overline{\tau}))_{L_2(K)}. \tag{4.17}$$

For $n - d = 2$ the coefficient of w_0 is as follows:

$$c_0 = (f, w_{01}(,\overline{\tau}))_{L_2(K)}. \tag{4.18}$$

§ 5. The Boundary-Value Problem in a Cone in the Scale of Weight Spaces

We introduce a scale of function spaces in accordance with the estimates (2.10) and (3.15). We denote by $DH_{\beta,q}(K;p)$ the space equipped with the norm

$$\|v; DH_{\beta,q}(K;p)\| = (\|\chi_p v; H^{q+2}_{\beta+q}(K;p)\|^2 + \gamma^2 \|v; H^{q+1}_{\beta+q}(K;p)\|^2)^{1/2} \tag{5.1}$$

for $n-d>2$ and with the norm

$$\|v;DH_{\beta,q}(K;p)\| = \Big(p^2\|r^{\beta-1}\chi_p v;L_2(K)\|^2 + \sum_{k=1}^{q+2}\|r^{\beta-2+k}\nabla^k(\chi_p v);L_2(K)\|^2 + \gamma^2\|v;H^{q+1}_{\beta+q}(K;p)\|^2\Big)^{1/2} \tag{5.2}$$

for $n-d=2$.

We introduce the space $RH_{\beta,q}(K;p)$ equipped with the norm

$$\|f;RH_{\beta,q}(K;p)\| = \Big(\sum_{j=0}^{q}(p/\gamma)^{2j}\|f;H^{q-j}_{\beta+q-j}(K;p)\|^2 + (p^{1-\beta+q}/\gamma^{q+1})^2\|f;L_2(K)\|^2\Big)^{1/2}. \tag{5.3}$$

For fixed p and γ the norm (5.3) is equivalent to the norm

$$(\|f;H^q_{\beta+q}(K;p)\|^2 + \|f;L_2(K)\|^2)^{1/2}. \tag{5.4}$$

In $RH_{\beta,q}(K;p)$, we introduce an unbounded operator $v\mapsto L(D_y,\zeta,\tau)v$ with domain $\overset{0}{\mathfrak{D}}_\beta$ if $n-d>2$ and $\overset{0}{\mathfrak{D}}_\beta(2)$ if $n-d=2$. This operator admits the closure which will be denoted by $A_{\beta,q}(\zeta,\tau)$.

The following assertion follows from the estimates (2.10) and (3.15).

Proposition 5.1 (cf. [2, Proposition 5.1]). *Let $\beta\leqslant 1$ for $n-d>2$ and $\beta<1$ for $n-d=2$. Let the line $\mathrm{Im}\lambda=\beta-1+(n-d-2)/2$ do not contain points of the spectrum of the pencil $\mathfrak{A}$. Then for any $q=0,1,2,\dots$ the following assertions hold.*

1. $DA_{\beta,q}(\zeta,\tau)\subset DA(\zeta,\tau)$.
2. $\mathrm{Ker}A_{\beta,q}(\zeta,\tau)=0$.
3. *The lineal* $\mathrm{Im}A_{\beta,q}(\zeta,\tau)$ *is closed in* $RH_{\beta,q}(K;p)$.

Our goal is to describe the lineal $\mathrm{Im}A_{\beta,q}(\zeta,\tau)$. Let $f\in RH_{\beta,q}(K;p)$, $f_n\in C_c^\infty(K)$, $f_n\to f$ in the $RH_{\beta,q}(K;p)$-norm. Let $L(D_y,\zeta,\tau)u_n=f_n$, $\partial_\nu u=0$ on ∂K, $u_n\in D(A(\zeta,\tau))$. Let β satisfy the assumptions of Proposition 5.1. We set

$$v_n = u_n - \chi\sum_{j\in J_\beta}\langle f_n, w_{-j}(,\overline{\tau})\rangle w_j, \tag{5.5}$$

where

$$J_\beta=\{j:(n-d-2)/2>\mathrm{Im}\lambda_j>\beta-1+(n-d-2)/2\}$$

if $n-d>2$, and

$$J_\beta=\{j:0>\mathrm{Im}\lambda_j>\beta-1\}\cup A$$

if $n-d=2$. Here, $A=\varnothing$ if $\beta>0$ and $A=\{0\}$ if $\beta\leqslant 0$. If $n-d=2$ and $\beta\leqslant 0$, then the coefficient at w_0 in (5.5) should be replaced with $\langle f_n, w_{01}(,\overline{\tau})\rangle$.

Lemma 5.2. *For any $q=0,1,\dots$ we have $v_n\in DA_{\beta,q}(\zeta,\tau)$.*

The proof is similar to the proof of Proposition 4.4 and is left to the reader.

From (5.5) it follows that

$$L(D_y,\zeta,\tau)v_n = f_n - \sum_{j=J_\beta} \langle f_n, w_{-j}(,\bar{\tau})\rangle L(D_y,\zeta,\tau)(\chi w_j). \qquad (5.6)$$

It is easy to see that the functionals $\langle f_n, w_{-j}(,\bar{\tau})\rangle$, $j \in J_\beta$, are continuous in the space $RH_{\beta,q}(K;p)$. Therefore, the right-hand side of (5.6) tends to the quantity

$$f - \sum_{j\in J_\beta} \langle f, w_{-j}(,\bar{\tau})\rangle L(D_y,\zeta,\tau)(\chi w_j)$$

in $RH_{\beta,q}(K;p)$ as $n\to\infty$. By Lemma 5.2 and the estimates (2.10) and (3.15), the sequence v_n converges to $v_0 \in DH_{\beta,q}(K;p)$ in the $DH_{\beta,q}(K;p)$-norm. Thus, for any function $f\in RH_{\beta,q}(K;p)$ we have

$$A_{\beta,q}(\zeta,\tau)v_0 = f - \sum_{j=J_\beta} \langle f, w_{-j}(,\bar{\tau})\rangle L(D_y,\zeta,\tau)(\chi w_j), \qquad (5.7)$$

where $v_0 \in DA_{\beta,q}(\zeta,\tau)$. Thus, we have proved the following assertion.

Proposition 5.3.

$$\mathrm{Im} A_{\beta,q}(K;p) = \{f \in RH_{\beta,q}(K;p) : \forall j \in J_\beta \langle f, w_{-j}(,\bar{\tau})\rangle = 0\}.$$

We recall that if $n-d=2$ and $\beta\leqslant 0$, then for $w_{-0}(,\bar{\tau})$ we take the function $w_{01}(,\bar{\tau})$.

Definition 5.4. By a *strong* (β,q)*-solution* to the problem (4.1) with the right-hand side $f\in RH_{\beta,q}(K;p)$ we mean a solution u to the equation

$$A_{\beta,q}(\zeta,\tau)u = f.$$

We summarize the results of this section.

Theorem 5.5. 1. *Let* $1\geqslant\beta>\mathrm{Im}\lambda_1+1-(n-d-2)/2$ *for* $n-d>2$, *and let* $1>\beta>\max\{0,\mathrm{Im}\lambda_1+1\}$ *for* $n-d=2$. *Then for every function* $f\in RH_{\beta,q}(K;p)$ *there exists a unique strong* (β,q)*-solution* u *to the problem* (4.1) *satisfying the following estimate*:

$$\|u; DH_{\beta,q}(K;p)\| \leqslant c(\beta,q)\|f; RH_{\beta,q}(K;p)\|. \qquad (5.8)$$

2. *Suppose that* $k\in\mathbb{N}$, $\beta\in(\mathrm{Im}\lambda_{k+1}+1-(n-d-2)/2,\ \mathrm{Im}\lambda_k+1-(n-d-2)/2)$, *and* $n-d>2$. *Then a strong* (β,q)*-solution exists only if* $f\in RH_{\beta,q}(K;p)$ *satisfies the condition*

$$\langle f, w_j(,\bar{\tau})\rangle = 0, \quad j=1,2,\ldots,k. \qquad (5.9)$$

A strong (β,q)*-solution is unique and satisfies the estimate* (5.8).

3. *Let* $n-d=2$, $\beta\in(\mathrm{Im}\lambda_{k+1}+1,\mathrm{Im}\lambda_k+1)$. *Consider two cases*:

(a) $\beta > 0$,

(b) $\beta \leqslant 0$.

In case (a), *a strong* (β, q)*-solution exists only if* f *satisfies the condition* (5.9). *In case* (b), *a solution exists only if* f *satisfies* (5.9) *and the following condition*:

$$\langle f, w_{01}(, \overline{\tau})\rangle = 0. \tag{5.10}$$

A strong (β, q)*-solution is unique and satisfies the estimate* (5.8).

4. *If* $f \in RH_{\beta,q}(K; p)$ *and a strong* (β, q)*-solution exists, then this solution is a strong solution to the problem* (4.1).

§ 6. Asymptotic Expansions of Solutions to the Problem in a Cone

We note that Lemma 5.2 remains valid if the functions w_j in (5.5) are replaced with partial sums of the series

$$w_j(y,\tau) = \Gamma(1+\nu_j)|y|^{i\lambda_j}\Phi_j(\omega)\sum_{m=0}^{\infty}\frac{(\tau^2-|\zeta|^2)^m(i|y|)^{2m}}{2^{2m}m!\Gamma(m+\nu_j+1)} \tag{6.1}$$

with sufficiently large number of terms (cf. (4.5) and [27, Sec. 7.2.2]). These partial sums will be denoted by $w_j^N(y,\tau)$ (N is the number of terms of the series (6.1) occurring in the partial sum). Thus, formula (5.7) remains valid if w_j is replaced with w_j^N for sufficiently large N.

To derive further estimates, it is convenient to deal with the normed parameter (ζ, τ) and consider the problem (2.2). By (5.7), if the right-hand side F of the problem

$$\begin{aligned} L(D_\eta, \theta)U &= F \quad \text{in } K, \\ \partial_\nu U &= 0 \quad \text{on } \partial K \end{aligned} \tag{6.2}$$

satisfies the condition $F \in RH_{\beta,q}(K;1)$, then the strong solution U to the problem (6.2) is represented in the form

$$U = \chi\sum_{j\in J_\beta}\langle F, w_{-j}(,\overline{\theta})\rangle w_j^N(,\theta) + V, \tag{6.3}$$

where V is a (β, q)-solution to the problem (6.2) with the right-hand side

$$F' := F - L(D_\nu,\theta)\Big\{\chi\sum_{j\in J_\beta}\langle F, w_{-j}(,\overline{\theta})\rangle w_j^N(,\theta)\Big\},$$

$$\theta = (\zeta/p, \tau/p), \quad \overline{\theta} = (\zeta/p, \overline{\tau}/p).$$

It is clear that

$$\|F'; RH_{\beta,q}(K;1)\| \leqslant c\|F; RH_{\beta,q}(K;1)\|,$$

where the constant c is independent of θ. Hence

$$\|V;DH_{\beta,q}(K;1)\| \leqslant c\|F;RH_{\beta,q}(K;1)\|. \tag{6.4}$$

We return to the variable $y=\eta/p$. We set

$$u(y)=U(py), \quad v(y)=V(py), \quad f(y)=p^2F(py).$$

The problem (6.2) becomes the problem (4.1). In the new variable, formula (6.3) takes the form

$$u(y)=\chi_p(y)\sum_{j=J_\beta}\langle f,w_{-j}(,\zeta,\bar{\tau})\rangle w_j^N(y,\zeta,\tau)+v(y) \tag{6.5}$$

(we used formulas (6.1), (4.4) and the relation $\lambda_j+\lambda_{-j}=i(n-d-2)$), where $\chi_p(y)=\chi(py)$.

It is easy to compute that

$$\|F;RH_{\beta,q}(K;1)\|^2=p^{n-d+2\beta-4}\|f;RH_{\beta,q}(K;p)\|^2, \tag{6.6}$$

$$\|V;DH_{\beta,q}(K;1)\|^2=p^{n-d+2\beta-4}\|v;DH_{\beta,q}(K;p)\|^2. \tag{6.7}$$

Thus, we have proved the following assertion.

Theorem 6.1. *Let β satisfy the assumptions of Proposition* 5.1, *and let $f\in RH_{\beta,q}(K;p)$. Then the strong solution u to the problem* (4.1) *is represented in the form*

$$u(y)=\chi_p(y)\sum_{j\in J_\beta}\langle f,w_{-j}(,\zeta,\bar{\tau})\rangle\Gamma(1+\nu_j)\sum_{m=0}^{N_j}\frac{(\tau^2-|\zeta|^2)^m(i|y|)^{2m}}{2^{2m}m!\Gamma(m+\nu_j+1)}\Phi_j(\omega)|y|^{i\lambda_j}+v(y), \tag{6.8}$$

where J_β is defined before Lemma 5.2, *N_j are sufficiently large integers, and v satisfies the estimate*

$$\|v;DH_{\beta,q}(K;p)\| \leqslant c\|f;RH_{\beta,q}(K;p)\|. \tag{6.9}$$

§ 7. The Problem in a Wedge

The results concerning the problem in a wedge is obtained by using the inverse Fourier transform from the assertions about the problem with parameters in a cone.

We consider the problem (0.1), (0.2) for $f\in V_0^0(Q;\gamma)$, $\gamma>0$ (cf. the definition of the norm in (1.5)). Let $\widehat{u}(,\zeta,\tau)$ be a strong solution to the problem (2.1). The function

$$u(y,z,t)=F^{-1}_{(\zeta,\tau)\to(z,t)}\widehat{u}(y,\zeta,\tau)$$

is called a *strong solution* to the problem (0.1), (0.2).

By Theorem 4.7, the following assertion holds.

Theorem 7.1. *For any $f \in V_0^0(q;\gamma)$ and $\gamma > 0$ there exists a unique strong solution u to the problem* (0.1), (0.2). *For $n-d>2$ we have*

$$\gamma \| u; V_0^1(Q;\gamma) \leqslant c \| f; V_0^0(q;\gamma) \|$$

and for $n-d=2$ the following inequality holds:

$$\gamma \| \exp(-\gamma t) \nabla_{(x,t)} u; L_2(Q) \| \leqslant \| f; V_0^0(Q;\gamma) \|.$$

The constant c is independent of $\gamma > 0$.

Let a function $\chi \in C_c^\infty(\mathbb{R}^{n-d})$ be equal to 1 in a neighborhood of the vertex O of the cone K. We introduce the operator

$$(Xu)(y,z,t) = F^{-1}_{(\zeta,\tau)\to(z,t)} \chi(py) F_{(z',t')\to(\zeta,\tau)} u(y,z',t').$$

We set

$$(\Lambda u)(y,z,t) = F^{-1}_{(\zeta,\tau)\to(z,t)} p F_{(z',t')\to(\zeta,\tau)} u(y,z',t').$$

For $\beta \in \mathbb{R}$, $\gamma > 0$, $q = 0,1,\dots$ we introduce the space $RV_{\beta,q}(Q;\gamma)$ equipped with the norm

$$\| f; RV_{\beta,q}(Q;\gamma) \| = \left(\sum_{j=0}^{q} \gamma^{-2j} \| \Lambda^j f; V^{q-j}_{\beta+q-j}(Q;\gamma) \|^2 + \gamma^{-2(1+q)} \| \Lambda^{1-\beta+q} f; V_0^0(Q;\gamma) \|^2 \right)^{1/2}.$$

The space $DV_{\beta,q}(Q:\gamma)$ is equipped with the norm

$$\| u; DV_{\beta,q}(Q;\gamma) \| = \left(\| Xu; V^{q+2}_{\beta+q}(Q;\gamma) \|^2 + \gamma^2 \| u; V^{q+1}_{\beta+q}(Q;\gamma) \|^2 \right)^{1/2}$$

for $n-d>2$ and the norm

$$\| u; DV_{\beta,q}(Q;\gamma) \|^2 = \int_{\mathbb{R}^{d+1}} \{ p^2 \| r^{\beta-1} \chi_p \widehat{u}(,\zeta,\tau); L_2(K) \|^2$$
$$+ \sum_{s=1}^{q+2} \| r^{\beta-2+s} \nabla^s (\chi_p \widehat{u}(,\zeta,\tau); L_2(K) \|^2 \} \, d\zeta \, d\sigma + \gamma^2 \| u; V^{q+1}_{\beta+q}(Q;\gamma) \|^2$$

for $n-d=2$. Let $\widehat{u}(,\zeta,\tau)$ be a strong (β,q)-solution to the problem (2.1). The function

$$u(y,z,t) = F^{-1}_{(\zeta,\tau)\to(z,t)} \widehat{u}(y,\zeta,\tau)$$

is called a *strong* (β,q)*-solution to the problem* (0.1), (0.2).

The following assertion is obtained from Theorem 5.5.

Theorem 7.2. 1. *Let $1 \geqslant \beta > \operatorname{Im}\lambda_1 + 1 - (n-d-2)/2$ if $n-d>2$, and let $1 > \beta > \max\{0, \operatorname{Im}\lambda_1 + 1\}$ if $n-d>2$. Let $\gamma > 0$, $q = 0,1,\dots$. Then for every function $f \in RV_{\beta,q}(Q,\gamma)$ there exists a unique strong (β,q)-solution u to the problem* (0.1), (0.2) *satisfying the following estimate*:

$$\| u; DV_{\beta,q}(Q;\gamma) \| \leqslant c(\beta,q) \| f; RV_{\beta,q}(Q;\gamma) \|, \tag{7.1.}$$

where the constant c is independent of $\gamma > 0$.

2. *Suppose that* $k \in \mathbb{N}$, $\beta \in (\operatorname{Im}\lambda_{k+1} + 1 + (n-d-2)/2, \operatorname{Im}\lambda_k + 1 + (n-d-2)/2)$, *and* $n-d > 2$. *Then a strong* (β, q)*-solution exists only if* $f \in RV_{\beta,q}(Q;\gamma)$ *satisfies the conditions*

$$\langle \widehat{f}(,\zeta,\tau), w_{-j}(,\zeta,\overline{\tau})\rangle = 0, \quad j = 1,2,\dots,k, \tag{7.2}$$

for all $\zeta \in \mathbb{R}^d$, $\tau = \sigma - i\gamma$. *A strong* (β,q)*-solution is unique and satisfies the estimate* (7.1).

3. *Suppose that* $n-d = 2$, $\beta \in (\operatorname{Im}\lambda_{k+1} + 1, \operatorname{Im}\lambda_k + 1)$. *Consider two cases*:

(a) $\beta > 0$,

(b) $\beta \leqslant 0$.

In case (a), *a strong* (β,q)*-solution exists only if* $f \in RV_{\beta,q}(Q;\gamma)$ *satisfies* (7.2). *In case* (b), *a strong* (β,q)*-solution exists only if f satisfies* (7.2) *and the condition* $\langle \widehat{f}(,\zeta,\tau), w_{01}(,\zeta,\overline{\tau})\rangle = 0$ *for all* $\zeta \in \mathbb{R}^d$, $\tau = \sigma - i\gamma$. *A strong* (β,q)*-solution is unique and satisfies* (7.1).

4. *Any strong* (β,q)*-solution is a strong solution. If there exist strong* (β,q)*- and* (β',q')*-solutions, then they coincide.*

Before the formulation of the theorem about the asymptotics of solutions to the problem (0.1), (0.2), we prove the following lemma.

Lemma 7.3. *The coefficient* $c_j(\zeta,\tau) = \langle f, w_{-j}(,\zeta,\overline{\tau})\rangle$ *in the asymptotic expansion* (6.8) *satisfies the estimate*

$$|c_j(\zeta,\tau)| \leqslant c\|f; RH_{\beta,q}(K;p)\| p^{\operatorname{Im}\lambda_{-j}+\beta-(n-d)/2}. \tag{7.3}$$

Proof. In the notation of Sec. 6, the following estimate holds:

$$|\langle F, w_{-j}(,\overline{\theta})\rangle| \leqslant c\|F; RH_{\beta,q}(K;1)\|. \tag{7.4}$$

Taking into account the relations (6.6) and (4.5) and the equality $f(y) = p^2 F(py)$, we obtain (7.3) from (7.4) by making the change of variables $\eta = py$. □

Lemma 7.3 and Theorem 6.1 imply the following assertion.

Theorem 7.4. *Let* $\beta \leqslant 1$ ($\beta < 1$*inthecasen-d=2*), *and let the line* $\operatorname{Im}\lambda = \beta - 1 + (n-d-2)/2$ *do not contain points of the spectrum of the pencil* $\mathfrak{A}$. *Let* $\gamma > 0$, $f \in RV_{\beta,q}(Q;\gamma)$. *Then a strong solution u to the problem* (0.1), (0.2) *is represented in the form*

$$u(y,z,t) = \sum_{j\in J_\beta} \Gamma(1+\nu_j)|y|^{i\lambda_j}\Phi_j(\omega) \sum_{m=0}^{N_j} \frac{(\partial_t^2 - \Delta_z)^m (i|y|)^{2m}}{2^{2m} m!\,\Gamma(m+\nu_j+1)} (X\check{c}_j)(y,z,t) + \check{v}(y,z,t), \tag{7.5}$$

where

$$\begin{gathered}\check{\varphi}(y,z,t) = F^{-1}_{(\zeta,\tau)\to(z,t)}\varphi(y,\zeta,\tau),\\ c_j(\zeta,\tau) = \langle \widehat{f}(,\zeta,\tau), w_{-j}(,\zeta,\overline{\tau})\rangle_{L_2(K)}\end{gathered} \tag{7.6}$$

and the function $(z,t) \mapsto \exp(-\gamma t)\check{c}_j(z,t) \equiv \check{c}_j^{\gamma}(z,t)$ *satisfies the inequality*

$$\|\check{c}_j^{\gamma}; H^{(n-d)/2-\operatorname{Im}\lambda_{-j}-\beta}(\mathbb{R}^{d+1})\| \leqslant c\|f; RH_{\beta,q}(Q;\gamma)\|. \tag{7.7}$$

The remainder $\check{v}(y,z,t)$ *in* (7.5) *satisfies the estimate*

$$\|\check{v}(y,z,t); DV_{\beta,q}(Q;\gamma)\| \leqslant c\|f; RH_{\beta,q}(Q;\gamma)\|. \tag{7.8}$$

The constant c *in* (7.7) *and* (7.6) *is independent of* $\gamma > 0$.

Remark 7.5. As is known (cf., for example, [20]), the operator X from formula (7.5) is the operator of smooth extension of functions defined on the edge $M \times \mathbb{R}$ (cf. 1.2) inside the wedge $Q = \mathcal{K} \times \mathbb{R}$. If $\check{c}_j$ is a sufficiently smooth function (say, β is a negative number with large modulus (cf. (7.6))), then the principal term in the jth summand in (7.5) can be written in the form $\check{c}_j(z,t)r^{i\lambda_j}\Psi_j(\omega)$.

§ 8. Explicit Formulas for the Coefficients of Asymptotics

With the special solution w_{-k} (w_{01}) to the homogeneous problem for the Helmholtz equation in the cone (cf. (4.3)) we associate the special solution

$$W_{-k} = F^{-1}_{(\zeta,\tau)\to(z,t)} w_{-k} \quad (W_{01} = F^{-1}_{(\zeta,\tau)\to(z,t)} w_{01})$$

to the homogeneous boundary-value problem for the wave equation in a wedge. The following lemma was established by the authors, together with S. I. Matyukevich.

Lemma 8.1. *Suppose that* $\mathbf{1}_{[s,+\infty)}(t)$ *is the characteristic function of the set* $[s,+\infty)$, $F(a,b,c,x)$ *is a hypergeometric function,* d/dt *is the generalized differentiation operation,*

$$N_k = N(d,\nu_k) = [d/2+\nu_k]+1, \quad \theta_k = \theta(d,\nu_k) = N_k - d/2 - \nu_k, \quad r = |y|, \quad \omega = y/|y|.$$

Then the following assertions hold.

1. *For* $d \geqslant 1$

$$\begin{aligned} W_{-k}(y,z,t) = {} & \frac{2^{3/2-\nu_k-\theta_k}\pi}{\Gamma(\nu_k)\Gamma(\theta_k+1/2)} r^{2\nu_k+i\lambda_{-k}}(|z|^2+r^2)^{-(\nu_k+\theta_k+d/2)/2} \\ & \times \Phi_k(\omega)\left(\frac{d}{dt}\right)^{N_k}\{\mathbf{1}_{[\sqrt{|z|^2+r^2},+\infty)}(t)(t^2-|z|^2-r^2)^{\theta_k-1/2} \\ & \times F((\theta_k-\nu_k)/2,(\theta_k+\nu_k)/2,\theta_k+1/2,1-t^2/(|z|^2+r^2))\}. \end{aligned} \tag{8.1}$$

2. *For* $d = 0$

$$\begin{aligned} W_{-k}(y,t) = {} & \frac{2^{3/2-\nu_k-\theta_k}\pi}{\Gamma(\nu_k)\Gamma(\theta_k+1/2)} r^{\nu_k-\theta_k+i\lambda_{-k}}\Phi_k(\omega)\left(\frac{d}{dt}\right)^{N_k}\{\mathbf{1}_{[r,+\infty)}(t)(t^2-r^2)^{\theta_k-1/2} \\ & \times F((\theta_k-\nu_k)/2,(\theta_k+\nu_k)/2,\theta_k+1/2,1-t^2/r^2)\}. \end{aligned} \tag{8.2}$$

3. *For* $d=2l$, $l=1,2,\ldots$,

$$W_{01}(y,z,t)=\alpha^{-1/2}(2\pi)^{1/2}(|z|^2+r^2)^{-l/2}\left(\frac{d}{dt}\right)^l\Big\{\mathbf{1}_{[\sqrt{|z|^2+r^2},+\infty)}(t)(t^2-r^2-|z|^2)^{-1}$$
$$\times\cosh[l\operatorname{arcosh}(t/\sqrt{|z|^2+r^2})]\Big\}. \tag{8.3}$$

4. *For* $d=2l+1$, $l=0,1,\ldots$,

$$W_{0,1}(y,z,t)=\alpha^{-1/2}\pi(|z|^2+r^2)^{-l/2-1/2}\left(\frac{d}{dt}\right)^{l+1}$$
$$\times\Big\{\mathbf{1}_{[\sqrt{|z|^2+r^2},+\infty)}(t)F(1/4,1/4,1,1-t^2/(|z|^2+r^2))\Big\}. \tag{8.4}$$

5. *For* $d=0$

$$W_{01}(y,t)=\alpha^{-1/2}(2\pi)^{1/2}\mathbf{1}_{[r,+\infty)}(t)(t^2-r^2)^{-1}. \tag{8.5}$$

Proof. By (4.4) and (4.7), we have

$$W_{-k}(y,z,t)=(2^{1-\nu_k}/\Gamma(\nu_k))r^{i\lambda-k}\Phi_k(\omega)S_{d,\nu_k}(r,z,t), \tag{8.6}$$

$$W_{01}(y,z,t)=\alpha^{-1/2}S_{d,0}(r,z,t), \tag{8.7}$$

where

$$S_{d,\nu}(r,z,t)=(2\pi)^{-(d+1)/2}\int\limits_{\operatorname{Im}\tau=-\gamma}d\tau\exp(i\tau t)\int\limits_{\mathbb{R}^d}\left(ir\sqrt{\tau^2-|\zeta|^2}\right)^\nu K_\nu\left(ir\sqrt{\tau^2-|\zeta|^2}\right)\exp(iz\zeta)\,d\zeta \tag{8.8}$$

for $d\geqslant 1$ and

$$S_{0,\nu}(r,t)=(2\pi)^{-1/2}\int\limits_{\operatorname{Im}\tau=-\gamma}\exp(i\tau t)(ir\tau)^\nu K_\nu(ir\tau)\,d\tau \tag{8.9}$$

for $d=0$.

For $d\geqslant 2$ we pass to the spherical coordinates in the interior integral in (8.8) (the north pole is located at the point $z/|z|$). We have

$$S_{d,\nu}(r,z,t)=(2\pi)^{-(d+1)/2}\int\limits_{-\infty-\gamma i}^{+\infty-\gamma i}d\tau\exp(it\tau)\int\limits_0^{+\infty}d\rho\rho^{d-1}\left(ir\sqrt{\tau^2-\rho^2}\right)^\nu K_\nu\left(ir\sqrt{\tau^2-\rho^2}\right)$$
$$\times\left[\int\limits_0^\pi\exp(i\rho|z|\cos\theta_1)\sin^{d-2}\theta_1\,d\theta_1\right]\left\{2\pi\prod_{k=1}^{d-3}\int\limits_0^\pi\sin^k\theta\,d\theta\right\}. \tag{8.10}$$

By [27, Sec. 7.12, formula (9)], the integral in the square brackets is equal to the quantity

$$2^{(d-2)/2}\sqrt{\pi}\Gamma((d-1)/2)J_{d/2-1}(\rho|z|)(\rho|z|)^{(2-d)/2},$$

where J_ν is the Bessel function of the first kind. The expression in the square brackets is equal to the quantity $2\pi^{(d-1)/2}/\Gamma((d-1)/2)$ (cf., for example, [28, Sec. 1.2, formula (9)]). Hence

$$S_{d,\nu}(r,z,t) = (2\pi)^{-1/2}|z|^{(2-d)/2} \int\limits_{\operatorname{Im}\tau=-\gamma} d\tau \exp(it\tau) \times \int\limits_0^{+\infty} \rho^{d/2}(ir\sqrt{\tau^2-\rho^2})^\nu K_\nu(ir\sqrt{\tau^2-\rho^2})J_{d/2-1}(\rho|z|)\,d\rho. \tag{8.11}$$

For $d=1$

$$S_{1,\nu}(r,z,t) = \pi^{-1} \int\limits_{\operatorname{Im}\tau=-\gamma} d\tau \exp(i\tau t) \int\limits_0^{+\infty} (ir\sqrt{\tau^2-\rho^2})^\nu K_\nu(ir\sqrt{\tau^2-\rho^2})\cos(\rho|z|)\,d\rho.$$

By [29, Sec. 5.8, formula 5.8.2], we have

$$\cos(\rho|z|) = (\pi\rho|z|/2)^{1/2}J_{-1/2}(\rho|z|).$$

Consequently, formula (8.11) is also valid for $d=1$. Making the change of variables $\tau = is$ in (8.11) and setting $\mu = -\nu$, we find

$$S_{d,\nu}(r,z,t) = (2\pi)^{-1/2}|z|^{(2-d)/2}r^\nu i \int\limits_{\gamma+i\infty}^{\gamma-i\infty} ds \exp(-st) \times \int\limits_0^{+\infty} \rho^{d/2}(s^2+\rho^2)^{-\mu/2}K_\mu\left(r\sqrt{s^2+\rho^2}\right)J_{d/2-1}(\rho|z|)\,d\rho. \tag{8.12}$$

The interior integral in (8.12) is the Sonin–Hegenbauer integral (cf. [27, Sec.14.2, formula (46)]) and is equal to

$$|z|^{d/2-1}r^\nu s^{d/2+\nu}(|z|^2+r^2)^{(-\nu-d/2)/2}K_{-\nu-d/2}(s\sqrt{|z|^2+r^2}).$$

Therefore, for $d \geqslant 1$ we have

$$S_{d,\nu}(r,z,t) = (2\pi)^{-1/2}i^{-1}r^{2\nu}(|z|^2+r^2)^{-\nu/2-d/4} \int\limits_{\operatorname{Re}s=\gamma} \exp(-st)s^{d/2+\nu}K_{\nu+d/2}\left(s\sqrt{|z|^2+r^2}\right)ds. \tag{8.13}$$

For $d=0$ the change of variables $s=i\tau$ in (8.9) leads to the equality

$$S_{0,\nu}(r,t) = (2\pi)^{-1/2}i^{-1}r^\nu \int\limits_{\operatorname{Re}s=\gamma} \exp(st)s^\nu K_\nu(rs)\,ds. \tag{8.14}$$

We set

$$N = N(d,\nu) = [d/2+\nu]+1, \quad \theta = \theta(d,\nu) = N(d,\nu)-d/2-\nu, \quad d=0,1,\dots.$$

Then (in the sense of distributions)

$$S_{d,\nu}(r,z,t) = (2\pi)^{1/2} r^{2\nu}(|z|^2+r^2)^{-\nu/2-d/4}\left(\frac{d}{dt}\right)^N \mathcal{L}_{s\to(-t)}\left(s^{-\theta}K_{\nu+d/2}\left(s\sqrt{|z|^2+r^2}\right)\right) \tag{8.15}$$

for $d \geqslant 1$ and

$$S_{0,\nu}(r,t) = (2\pi)^{1/2} r^{\nu}\left(\frac{d}{dt}\right)^N \mathcal{L}_{s\to(-t)}(s^{-\theta}K_\nu(sr)) \tag{8.16}$$

for $d=1$. For $\nu = 0$, $d = 2l$, $l = 1,2,\dots$, we have

$$S_{d,0}(r,z,t) = (2\pi)^{1/2}(|z|^2+r^2)^{-l/2}\left(\frac{d}{dt}\right)^l \mathcal{L}_{s\to(-t)}\left(K_l\left(s\sqrt{|z|^2+r^2}\right)\right). \tag{8.17}$$

For $\nu = 0$, $d = 2l+1$, $l = 0,1,\dots$, we have

$$S_{d,0}(r,z,t) = (2\pi)^{1/2}(|z|^2+r^2)^{-l/2-1/4}\left(\frac{d}{dt}\right)^{l+1} \mathcal{L}_{s\to(-t)}\left(s^{-1/2}K_{l+1/2}\left(s\sqrt{|z|^2+r^2}\right)\right). \tag{8.18}$$

For $\nu = 0$, $d = 0$ we have

$$S_{0,0}(r,t) = (2\pi)^{1/2}\mathcal{L}_{s\to(-t)}(K_0(sr)), \tag{8.19}$$

where

$$\mathcal{L}_{s\to t}u(s) = (2\pi i)^{-1}\int_{\gamma-i\infty}^{\gamma+i\infty} \exp(st)u(s)\,ds$$

denotes the inverse Laplace transform. The Laplace transforms in (8.15)–(8.19) were computed in [30, Sec. 5.15, formulas (12) and (15)]. Namely,

$$\begin{aligned}\mathcal{L}_{s\to(-t)}\left(s^{-\theta}K_{\nu+d/2}\left(s\sqrt{|z|^2+r^2}\right)\right) &= \mathbf{1}_{[\sqrt{|z|^2+r^2},+\infty)}(t)(2^{-\theta}/\Gamma(\theta+1/2))\pi^{1/2}(|z|^2+r^2)^{-\theta/2}\\ &\times (t^2-|z|^2-r^2)^{\theta-1/2}F(\theta/2-\nu/2,\theta/2+\nu/2,\theta+1/2,1-t^2/(|z|^2+r^2)),\end{aligned} \tag{8.20}$$

$$\begin{aligned}\mathcal{L}_{s\to(-t)}(s^{-\theta}K_\nu(sr)) &= \mathbf{1}_{[r,+\infty)}(t)(2^{-\theta}/\Gamma(\theta+1/2))\pi^{1/2}r^{-\theta}(t^2-r^2)^{\theta-1/2}\\ &\times F(\theta/2-\nu/2,\theta/2+\nu/2,\theta+1/2,1-t^2/r^2),\end{aligned} \tag{8.21}$$

$$\begin{aligned}\mathcal{L}_{s\to(-t)}\left(K_l\left(s\sqrt{|z|^2+r^2}\right)\right) &= \mathbf{1}_{[\sqrt{|z|^2+r^2},+\infty)}(t)(t^2-r^2-|z|^2)^{-1}\\ &\times \cosh[l\,\mathrm{arcosh}(t/(|z|^2+r^2)^{1/2}],\end{aligned} \tag{8.22}$$

$$\begin{aligned}\mathcal{L}_{s\to(-t)}\left(s^{-1/2}K_{l+1/2}\left(s\sqrt{|z|^2+r^2}\right)\right) &= \mathbf{1}_{[\sqrt{|z|^2+r^2},+\infty)}(t)(\pi/2)^{1/2}(|z|^2+r^2)^{-1/4}\\ &\times F(1/4,1/4,1,1-t^2/(|z|^2+r^2)),\end{aligned} \tag{8.23}$$

$$\mathcal{L}_{s\to(-t)}(K_0(sr)) = \mathbf{1}_{[1,+\infty)}(t)(t^2-r^2)^{-1}. \tag{8.24}$$

It remains to use (8.6) and (8.7). □

Remark 8.2. From the formulas of Lemma 8.1 it follows that

(1) $W_{-k}(y,z,t)=0$ if $t<\sqrt{|y|^2+|z|^2}$,

(2) $\operatorname{sing\,supp} W_k \subset \{(y,z,t)\in\mathbb{R}^{n+1} : t=\sqrt{|y|^2+|z|^2}\}$.

The same assertions are valid for W_{01}.

Owing to Remark 8.2, we can obtain some additional information about the coefficients

$$\check{c}_j(z,t) = \int\limits_{\mathbb{R}^d} dz_1 \int\limits_{\mathbb{R}} dt_1 \int\limits_{K} dy f(y,z_1,t_1) W_{-j}(y,z-z_1,t-t_1) \tag{8.25}$$

of the asymptotic expansion (7.5) obtained in Theorem 7.4.

Proposition 8.3. 1. *For the coefficients $\check{c}_j(z,t)$ of the asymptotics the "leading edge effect" holds*:

$$t<\inf\{|x|+s : (x,s)\in\operatorname{supp} f\} \Longrightarrow \check{c}_j(z,t)=0.$$

2. *If the singular support of the right-hand side f is bounded with respect to spatial variables and semibounded from above with respect to the time variable, then the coefficients $\check{c}_j(\ )$ are infinitely smooth functions in the domain* (*cf.* (7.7))

$$\{(z,t)\in\mathbb{R}^{d+1} : t>\sup\{t_1+\sqrt{|y|^2+|z-z_1|^2};\quad (y,z_1,t_1)\in\operatorname{sing\,supp} f\}.$$

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