PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 145, Number 9, September 2017, Pages 3915–3928
http://dx.doi.org/10.1090/proc/13494
Article electronically published on April 12, 2017

ISOSPECTRALITY, COMPARISON FORMULAS FOR DETERMINANTS OF LAPLACIAN AND FLAT METRICS WITH NON-TRIVIAL HOLONOMY

LUC HILLAIRET AND ALEXEY KOKOTOV

(Communicated by Ken Ono)

ABSTRACT. We study comparison formulas for ζ-regularized determinants of self-adjoint extensions of the Laplacian on flat conical surfaces of genus $g \geq 2$. The cases of trivial and non-trivial holonomy of the metric turn out to differ significantly.

1. INTRODUCTION

Let X be a compact Riemann surface of genus $g \geq 2$ and let $D = P_1 + \cdots + P_{2g-2}$ be a positive divisor of degree $2g-2$ on X. For simplicity, we consider only the generic situation: all the points P_k are assumed to be distinct. Then, due to the Troyanov theorem ([9]), there exists a unique (up to rescaling) conformal metric $\mathbf{m}$ on X that is flat with conical singularities of angle 4π at $P_1, \dots, P_{2g-2}$. If the divisor D belongs to the canonical class, then there also exists a holomorphic one form ω on X with simple zeros at $P_1, \dots, P_{2g-2}$. In that situation we thus obtain

$$\mathbf{m} = |\omega|^2, \tag{1.1}$$

so that the Troyanov metric $\mathbf{m}$ has trivial holonomy: i.e. the parallel transport along any loop that avoids the conical singularities of X maps any tangent vector to X to itself. Indeed, after the parallel transport along such a loop, a tangent vector has turned for an angle $2\pi k$ with $k \in \mathbb{Z}$. On the other hand, any flat metric with conical singularities that has trivial holonomy is of the form (1.1). It follows that the Troyanov metric $\mathbf{m}$ corresponding to the divisor D and conical angles 4π has trivial holonomy if and only if the divisor D belongs to the canonical class.

Consider now a general flat metric $\mathbf{m}$ with conical singularities of angle 4π, and define the Laplace operator $\Delta^{\mathbf{m}} : L^2(X, \mathbf{m}) \to L^2(X, \mathbf{m})$ that is associated with it. This operator is initially defined on the space $C_0^\infty(X \setminus \{P_1, \dots, P_{2g-2}\})$ of smooth functions that vanish in a neighbourhood of the conical points. The closure of $\Delta^{\mathbf{m}}$ is a symmetric operator in $L^2(X, \mathbf{m})$ with deficiency indices $(6g-6, 6g-6)$. It thus has infinitely many self-adjoint extensions. Among these, the Friedrichs extension, Δ_F, is extensively studied. Explicit formulas for the (modified: i.e. with zero modes excluded) ζ-regularized determinant $\det^* \Delta_F$ are found in [8] (in the case of trivial holonomy) and [7] (in the general case).

Received by the editors November 15, 2015 and, in revised form, September 25, 2016.
2010 *Mathematics Subject Classification.* Primary 30F30, 30F45, 35P99; Secondary 58J52, 30F10, 32G15.

The domain of the Friedrichs extension may include only functions that are bounded near the conical singularities. As a consequence, when the metric $\mathbf{m}$ has trivial holonomy, the function

$$u = \frac{\psi}{\omega}\,, \tag{1.2}$$

where ω is the holomorphic one form that defines the metric $\mathbf{m} = |\omega|^2$ and ψ is an arbitrary holomorphic one form on X not proportional to ω, belongs to $L^2(X, \mathbf{m})$, satisfies the equation $\Delta^{\mathbf{m}} u = 0$ in the classical sense outside the singularities $P_1, \dots, P_{2g-2}$, but *does not* belong to the domain of the Δ_F. It turns out that there exists another natural s.a. extension of $\Delta^{\mathbf{m}}$ that we denote by $\Delta_{\rm hol}$ whose domain does contain the functions of the form (1.2). Choosing an s.a. extension is equivalent to prescribing a certain kind of asymptotic behaviour near the conical points. For instance, functions in $\mathrm{dom}(\Delta_F)$ have to be bounded but can have both holomorphic and antiholomorphic terms in the asymptotic expansion near the conical points. On the contrary the functions in $\mathrm{dom}(\Delta_{\rm hol})$ are not necessarily bounded, but may only have holomorphic terms in this expansion (hence the name; see section 2.1).

In [5] was found a general comparison formula relating $\det^* \Delta_F$ to the determinant of any (regular) s.a. extension of the Laplacian for any flat conformal conical metric on X (with arbitrary conical angles). The main player in this comparison formula is the so-called S-matrix, $S(\lambda)$, of the conical surface X. The goal of this paper is twofold. First we will apply the results of [5] to relate the modified zeta-regularized determinant $\det^*(\Delta_{\rm hol})$ with the determinant of the Friedrichs extension. Then we will study in full detail the S-matrix in the special case of metrics with conical angles 4π: we will find explicit expressions for the entries of $S(\lambda)$ and their derivatives at $\lambda = 0$ through invariant holomorphic objects related to the Riemann surface X. This will result in a nice explicit expression for the determinant of the holomorphic s.a. extension. The following theorem illustrates our result in genus 2.

Theorem 1. *Let X be a Riemann surface of genus 2. For any two points P_1 and P_2 on X, let $\mathbf{m}$ be the Troyanov flat metric with conical singularities of angle 4π at P_1 and P_2 and let Δ_F and $\Delta_{\rm hol}$ be the corresponding s.a. extensions of $\Delta^{\mathbf{m}}$.*

If the divisor $P_1 + P_2$ is not in the canonical class or, equivalently, if $\mathbf{m}$ does not have trivial holonomy, then

$$\det{}^*\Delta_{hol} = c_2 \begin{vmatrix} B(P_1,P_1) & B(P_1,P_2) \\ B(P_2,P_1) & B(P_2,P_2) \end{vmatrix} \det{}^*\Delta_F\,, \tag{1.3}$$

where $B(P_i, P_j)$ is the Bergman reproducing kernel for the holomorphic differentials on X which is evaluated at P_i, P_j using distinguished holomorphic local parameters. The constant c_2 is universal.

If the divisor $P_1 + P_2$ is in the canonical class or, equivalently, if $\mathbf{m}$ has trivial holonomy, then

$$\det{}^*(\Delta_{\rm hol}) \;=\; \det{}^*(\Delta_F). \tag{1.4}$$

In the trivial holonomy case, it was previously observed by the first author ([4]) that the operators Δ_F and $\Delta_{\rm hol}$ are DtN isospectral (see the precise Definition 2 below). This immediately implies that their ζ-regularized determinants coincide.

It should be noted that when the divisor $P_1 + P_2$ enters the canonical class the matrix

$$\begin{pmatrix} B(P_1, P_1) & B(P_1, P_2) \\ B(P_2, P_1) & B(P_2, P_2) \end{pmatrix}$$

becomes degenerate and the dimension of the kernel of Δ_{hol} gets larger (by one unit).

Relation (1.4), together with the comparison formula from [5] and the explicit information about the entries of $S(\lambda)$ obtained in the present paper, leads to identity (4.7) below, which presents the second main result of the paper. The latter identity seems to be rather nontrivial, and we failed to find any direct way to prove it.

We make a certain effort to make it possible to read this paper independently of [5] (which is more technically involved): in our case of conical singularities of angle 4π the results and notation from [5] can be significantly simplified. We will briefly remind the reader of all the constructions and main ideas from [5], using this opportunity to make the presentation more transparent.

2. S-matrix of Euclidean surface with conical singularities

In this section we briefly recall the setting of the paper [5], adopting it to our (less general) situation.

2.1. **Self-adjoint extensions of the Laplacian.** Let X be a Riemann surface of genus $g \geq 2$ and let $\mathbf{m}$ be a flat conformal metric on X with conical singularities $P_1, \dots, P_{2g-2}$ of conical angles 4π. Let Δ be the closure of the Laplace operator (corresponding to $\mathbf{m}$) with domain

$$C_0^\infty(X \setminus \{P_1, \dots, P_{2g-2}\}) \subset L_2(X, \mathbf{m})$$

and let Δ^* be its adjoint operator. Let ξ_k be the distinguished holomorphic local parameter in a vicinity of P_k, i.e.

$$\mathbf{m} = 4|\xi_k|^2 |d\xi_k|^2$$

near P_k. Near any P_k, any $u \in \mathrm{dom}(\Delta^*)$ has an asymptotic expansion

$$\begin{aligned} u(\xi_k, \bar{\xi}_k) &= a_k(u) \frac{i}{\sqrt{2\pi}} \log|\xi_k| + b_k(u) \frac{1}{2\sqrt{\pi}} \frac{1}{\xi_k} + c_k(u) \frac{1}{2\sqrt{\pi}} \frac{1}{\bar{\xi}_k} \\ &\quad + \frac{i}{\sqrt{2\pi}} d_k(u) + \frac{1}{2\sqrt{\pi}} e_k(u) \xi_k + \frac{1}{2\sqrt{\pi}} f_k(u) \bar{\xi}_k \\ &\quad + o(|\xi|). \end{aligned} \tag{2.1}$$

The factors $\frac{1}{2\sqrt{\pi}}$, etc., are introduced in order to get the standard Darboux basis (see formula (2.2) below).

Let $u, v \in \mathcal{D}(\Delta^*)$. Then the Green formula implies the relation

$$\begin{aligned} \Omega([u],[v]) &:= \langle \Delta^* u, \bar{v} \rangle - \langle u, \Delta^* \bar{v} \rangle \\ &= \sum_{k=1}^{2g-2} X_k(u) \begin{pmatrix} 0 & -I_3 \\ I_3 & 0 \end{pmatrix} X_k(v)^t, \end{aligned} \tag{2.2}$$

where $X_k(u) = (a_k(u), b_k(u), \dots, f_k(u))$ and Ω is the symplectic form on the factor space $\mathrm{dom}(\Delta^*)/\mathrm{dom}(\Delta)$.

The self-adjoint extensions of Δ are in one-to-one correspondence with the Lagrangian subspaces of $\mathrm{dom}(\Delta^*)/\mathrm{dom}(\Delta)$; the Friedrichs extension corresponds to the Lagrangian subspace

$$a_k(u) = b_k(u) = c_k(u) = 0,\ k = 1, \dots, 2g-2\,;$$

the holomorphic extension corresponds to the Lagrangian subspace

$$a_k(u) = c_k(u) = f_k(u) = 0,\ k = 1, \dots, 2g-2\,.$$

2.2. **Special solutions and S-matrix.** For any $s\in\mathbb{R}$, denote $H^s_F(X):=\mathrm{dom}(\Delta_F^{\frac{s}{2}})$. Since X is compact, the spectrum of Δ_F is discrete and consists only of finite multiplicity eigenvalues. Let $(\phi_\ell)_{\ell\geq 0}$ be a real orthonormal basis of eigenfunctions for Δ_F and denote by λ_ℓ the eigenvalue that corresponds to ϕ_ℓ. This basis will be fixed throughout the paper.

We define the linear functionals $\Lambda_{\xi_k}, \Lambda_{\bar\xi_k}$ and Λ_{0k} on $H^2_F(X)$ in such a way that any $u \in H^2_F(X)(:= \mathrm{dom}(\Delta_F))$ has the following asymptotic expansion near P_k:

$$u(\xi_k, \bar\xi_k) \;=\; \frac{i}{\sqrt{\pi}}\Lambda_{0k}(u) \;+\; \frac{1}{2\sqrt{\pi}}\Lambda_{\xi_k}(u)\cdot\xi_k \;+\; \frac{1}{2\sqrt{\pi}}\Lambda_{\bar\xi_k}(u)\cdot\bar\xi_k \;+\; o(|\xi_k|).$$

The Λ_{ξ_k} thus appears as (continuous) linear functionals on $H^2_F(X)$. It follows that the sequence $\left(\frac{\Lambda_{\xi_k}(\phi_\ell)}{\lambda_\ell-\lambda}\right)_{\ell\geq 0}$ is in $\ell^2(\mathbb{N},\mathbb{C})$.

Let λ not belong to the spectrum Δ_F. By the functional calculus, the resolvent $(\Delta_F - \lambda)^{-1}$ is a one-to-one map from $H^s_F(X)$ to $H^{s+2}_F(X)$ for any $s \in \mathbb{R}$ (where we recall that, by definition, $H^s_F(X) := \mathrm{dom}(\Delta_F^{\frac{s}{2}})$).

Since Λ_{ξ_k} is a continuous linear functional on H^2_F (i.e. an element of $H^{-2}_F(X)$), we may thus define $G_{\xi_k} \in L^2(X)$ by

$$G_{\xi_k} := (\Delta_F - \lambda)^{-1}\,\Lambda_{\xi_k},$$

and the functions $G_{\bar\xi_k},\ G_{0k}$ are obtained by a similar formula.

By definition, we have

$$\forall k,\ G_{\xi_k}(\cdot,\lambda) \;=\; \sum_{\ell\geq 0}\frac{\Lambda_{\xi_k}(\phi_\ell)}{\lambda_\ell - \lambda}\phi_\ell. \tag{2.3}$$

There are similar expressions for $G_{\bar\xi_k}$.

We recall that G_{ξ_k} can be constructed in the following way. Start with $F_{\xi_k} := \frac{1}{2\sqrt{\pi}}\cdot\frac{1}{\xi_k}\chi(\xi_k,\bar\xi_k)$ where χ is a cutoff function that is identically one near P_k. We then have

$$G_{\xi_k} = F_{\xi_k} - (\Delta_F - \lambda)^{-1}(\Delta^* - \lambda)F_{\xi_k}.$$

Thus, by construction, $G_{\xi_k} - F_{\xi_k}$ belongs to $H^2_F(X)$ and so does $\partial_\lambda G_{\xi_k}$. The latter has the following (convergent in $L^2(X)$) expression in the chosen orthonormal basis:

$$\partial_\lambda G_{\xi_k}(\cdot,\lambda) \;=\; \sum_{\ell\geq 0}\frac{\Lambda_{\xi_k}(\phi_\ell)}{(\lambda_\ell - \lambda)^2}\phi_\ell. \tag{2.4}$$

By evaluating the linear functionals Λ on $G_{\xi_k} - F_{\xi_k}$, we obtain a $(6g-6)\times(6g-6)$ matrix $S(\lambda)$.

In what follows we will often use the following convenient and self-explanatory notation: the entries of S will be denoted by expressions of the type $S^{\bar\xi_k\xi_j}(\lambda)$ that will correspond to the coefficient of ξ_j in the expansion of $G_{\bar{\xi}_k}$. Thus, by definition, near P_k we will have

$$G_{\xi_k} = \frac{1}{2\sqrt{\pi}}\frac{1}{\xi_k} + \frac{i}{\sqrt{2\pi}}S^{\xi_k 0_k}(\lambda) + \frac{1}{2\sqrt{\pi}}S^{\xi_k\xi_k}(\lambda)\xi_k + \frac{1}{2\sqrt{\pi}}S^{\xi_k\bar{\xi}_k}(\lambda)\bar{\xi}_k + o(|\xi_k|)$$

as $\xi_k \to 0$, and near P_j we will have

$$G_{\xi_k} = \frac{i}{\sqrt{2\pi}}S^{\xi_k 0_j}(\lambda) + \frac{1}{2\sqrt{\pi}}S^{\xi_k\xi_j}(\lambda)\xi_j + \frac{1}{2\sqrt{\pi}}S^{\xi_k\bar{\xi}_j}(\lambda)\bar{\xi}_j + o(|\xi_j|)$$

as $\xi_j \to 0$.

It is convenient to represent this S-matrix with a block structure

$$S(\lambda) = \begin{pmatrix} S_{00}(\lambda) & S_{0a}(\lambda) & S_{0h}(\lambda) \\ S_{a0}(\lambda) & S_{aa}(\lambda) & S_{ah}(\lambda) \\ S_{h0}(\lambda) & S_{ha}(\lambda) & S_{hh}(\lambda) \end{pmatrix},$$

where the indices a and h stand for holomorphic and anti-holomorphic. Thus, for instance, the S_{ah} block records the coefficients of the anti-holomorphic part of the G_{ξ_k}:

$$S_{ah}^{jk} = S^{\bar{\xi}_j\xi_k}.$$

The S-matrix is holomorphic outside the spectrum of Δ_F. The blocks S_{00}, S_{0a}, S_{0h}, S_{h0} and S_{a0} blow up as $\lambda \to 0$, whereas the blocks S_{aa}, S_{ah}, S_{ha} and S_{hh} are regular at $\lambda = 0$.

In the next subsection we find explicit expressions for $S_{aa}(0)$, $S_{ah}(0)$, $S_{ha}(0)$ and $S_{hh}(0)$. These expressions present the first new result of this paper.

2.3. $S(0)$ through Schiffer and Bergman kernels.

2.3.1. *Some kernels from complex geometry.* Choose a marking for the Riemann surface X, i.e. a canonical basis $a_1, b_1, \ldots, a_g, b_g$ of $H_1(X, \mathbf{Z})$. Let $\{v_1, \ldots, v_g\}$ be the basis of holomorphic differentials on X that is normalized via

$$\int_{a_i} v_j = \delta_{ij}\,.$$

Then the matrix of b-periods of the marked Riemann surface X is defined as

$$\mathbb{B} = [b_{ij}]_{i,j\le g}, \text{ with } b_{ij} := \int_{b_i} v_j\,.$$

Let $W(\,\cdot\,,\,\cdot\,)$ be the canonical meromorphic bidifferential on $X \times X$ (see [3]). Recall that it is symmetric, $(W(P,Q) = W(Q,P))$, and normalized by

$$\int_{a_i} W(\,\cdot\,, P) = 0.$$

Recall also the following identities:

$$\int_{b_j} W(\,\cdot\,, P) = 2\pi i v_j(P).$$

The only pole of the bidifferential W is a double pole along the diagonal $P = Q$. In any holomorphic local parameter $x(P)$, the following asymptotic expansion near the diagonal holds:

$$W(x(P), x(Q)) = \left(\frac{1}{(x(P)-x(Q))^2} + H(x(P), x(Q))\right) dx(P)dx(Q), \tag{2.5}$$

where

$$H(x(P), x(Q)) = \frac{1}{6} S_B(x(P)) + O(x(P) - x(Q)),$$

as $Q \to P$, with $S_B(\cdot)$ the Bergman projective connection.

The Schiffer bidifferential (or Schiffer kernel) is defined by

$$\mathcal{S}(P, Q) = W(P, Q) - \pi \sum_{i,j} (\Im \mathbb{B})^{-1}_{ij} v_i(P) v_j(Q),$$

and the Schiffer projective connection, S_{Sch}, by the asymptotic expansion

$$\mathcal{S}(x(P), x(Q)) \\ = \left(\frac{1}{(x(P) - x(Q))^2} + \frac{1}{6} S_{\mathrm{Sch}}(x(P)) + O(x(P) - x(Q)) \right) dx(P) dx(Q) .$$

This implies the equality

$$S_{\mathrm{Sch}}(x) = S_B(x) - 6\pi \sum_{i,j} (\Im \mathbf{B})^{-1}_{ij} v_i(x) v_j(x) . \tag{2.6}$$

In contrast to the canonical meromorphic differential and the Bergman projective connection, the Schiffer bidifferential and the Schiffer projective connection are independent of the marking of the Riemann surface X.

We will also need the Bergman kernel which is the reproducing kernel for holomorphic differentials on X. Its expression is given by

$$B(x, \bar{x}) = \sum_{ij} (\Im \mathbb{B})^{-1}_{ij} v_i(x) \overline{v_j(x)} . \tag{2.7}$$

2.3.2. *Harmonic functions on X.* Let X be a compact Riemann surface of genus $g > 1$ provided with a flat conformal metric m with $2g - 2$ conical singularities at $P_1, \dots, P_{2g-2}$ of conical angles 4π. We do not assume that this metric has trivial holonomy: the divisor class $P_1 + \dots + P_{2g-2}$ is not necessarily canonical.

The harmonic functions on $X \setminus \{P_k\}$ which are square-integrable are uniquely determined by their singular behaviour near the points P_k. Thus, there exists a unique harmonic function H_k with the asymptotic behaviour

$$H_k(\xi_k, \bar{\xi}_k) = \frac{1}{\xi_k} + a_{kk} + b_{kk}\xi_k + c_{kk}\bar{\xi}_k + o(|\xi_k|) \quad \text{at} \quad P_k,$$

$$H_k(\xi_j, \bar{\xi}_j) = a_{kj} + b_{kj}\xi_j + c_{kj}\bar{\xi}_j + o(|\xi_j|) \;\; \text{at} \;\; P_j; \;\; j \neq k$$

(we remind the reader that ξ_k denotes the distinguished local parameter for $\mathbf{m}$ near P_k). Comparing with the definition of G_{ξ_k} we see that

$$H_k = 2\sqrt{\pi} G_{\xi_k},$$

and thus

$$b_{kj} = S^{\xi_k \xi_j}(0),$$
$$c_{kj} = S^{\xi_k \bar{\xi}_j}(0).$$

Notice that $S^{\bar{\xi}_k \bar{\xi}_j}(0) = \overline{S^{\xi_k \xi_j}(0)}$ and $S^{\bar{\xi}_k \xi_j}(0) = \overline{S^{\xi_k \bar{\xi}_j}(0)}$ and, therefore, the $S(0)$ is completely known once we find b_{kj} and c_{kj}.

Proposition 1. *The following relations hold:*

$$
\begin{aligned}
S^{\xi_k\xi_k} &= -\frac{1}{6}S_{\rm Sch}(\xi_k)\Big|_{\xi_k=0}\,,\\
S^{\xi_k\xi_j} &= -\mathcal{S}(P_k,P_j)\,, k\neq j\,,\\
S^{\xi_k\bar\xi_j} &= -\pi B(P_k,P_j)\,,
\end{aligned}
$$

where the values of (bi)differentials at the points P_i are taken w.r.t. the distinguished local parameter ξ_i.

Proof. Introduce the one forms Ω_k and Σ_k on X:

$$\Omega_k=-W(\,\cdot\,,P_k)+2\pi i\sum_{\alpha,\beta}(\Im\mathbb{B})^{-1}_{\alpha\beta}\,\{\Im v_\beta(P_k)\}\,v_\alpha(\cdot)\,,$$

$$\Sigma_k=-iW(\,\cdot\,,P_k)+2\pi i\sum_{\alpha,\beta}(\Im\mathbb{B})^{-1}_{\alpha\beta}\,\{\Re v_\beta(P_k)\}\,v_\alpha(\cdot)\,,$$

where

$$v_\beta(P_k):=v_\beta(\xi_k)|_{\xi_k=0}\,.$$

All the periods of the differentials Ω_k and Σ_k are purely imaginary; therefore, the following expression correctly defines a function H_k on X:

$$H_k(Q)=\Re\left\{\int_{P_0}^Q\Omega_k\right\}-i\Re\left\{\int_{P_0}^Q\Sigma_k\right\} \tag{2.8}$$

where $P_0\neq P_k$ is an arbitrary base point.

Simple calculation shows that

$$H_k(\xi_k,\bar\xi_k)=\frac{1}{\xi_k}+\text{const}+\left[-\frac{1}{6}S_B(P_k)+\pi\sum_{\alpha,\beta}(\Im\mathbb{B})^{-1}_{\alpha\beta}v_\alpha(P_k)v_\beta(P_k)\right]\xi_k$$

$$+\left[-\pi\sum_{\alpha\beta}(\Im\mathbb{B})^{-1}_{\alpha\beta}\bar v_\alpha(P_k)v_\beta(P_k)\right]\bar\xi_k+o(|\xi_k|)$$

and

$$H_k(\xi_j,\bar\xi_j)=\text{const}+\left[-W(P_k,P_j)+\pi\sum_{\alpha\beta}(\Im\mathbb{B})^{-1}_{\alpha\beta}v_\beta(P_k)v_\alpha(P_j)\right]\xi_j$$

$$+\left[-\pi\sum_{\alpha\beta}(\Im\mathbb{B})^{-1}_{\alpha\beta}v_\beta(P_k)\bar v_\alpha(P_j)\right]\bar\xi_j+o(|\xi_j|)$$

for $j\neq k$, which implies the proposition. ∎

3. Bergman kernel and holomorphic one-forms

Let $B(\cdot\,,\,\cdot)$ be the Bergman reproducing kernel for holomorphic differentials (2.7) on X, let $P_1,\dots,P_{2g-2}$ be 2g-2 distinct points of X and let ξ_k be a holomorphic local parameter at P_k. Introduce the $(2g-2)\times(2g-2)$ matrix $\mathcal{B}$:

$$\mathcal{B}=[B(P_j,P_k)]_{j,k=1,\dots,2g-2}\ ,$$

where the value of B at P_k is taken with respect to the local parameter ξ_k. We observe that the rank of $\mathcal{B}$ is independent of the choice of the local parameter.

According to Proposition 1 we have

$$\mathcal{B}=-\frac{1}{\pi}S_{ha}(0).$$

It follows that knowing the S matrix implies knowing the properties of the Riemann surface that are encoded in $\mathcal{B}$. We also have the following proposition.

Proposition 2. *The following two statements are equivalent.*

(A) *The divisor $P_1+\dots+P_{2g-2}$ coincides with the divisor of some holomorphic one-form ω.*
(B) $\operatorname{rank}\mathcal{B}<g$.

Before proving this proposition, we recall the following definition.

Definition 1. On a Riemann surface X of genus g, a divisor $Q_1+Q_2+\dots+Q_g$ is called *special* when

$$\begin{aligned}&i(Q_1+\dots+Q_g)\\&\qquad:=\dim\{\omega\text{-holomorphic one form: }(\omega)>Q_1+\dots+Q_g\ \text{ or }\ \omega=0\}>0\,.\end{aligned}$$

Equivalently, the divisor $Q_1+\dots+Q_g$ is special iff

$$\det\left[v_j(Q_k)\right]_{1\le j,k\le g}=0. \tag{3.1}$$

Proof. $(\mathbf{A})\Rightarrow(\mathbf{B})$: Since the positive divisor $P_1+\dots+P_{2g-2}$ is in the canonical class, for any $i_1,\dots,i_g$, $1\le i_1<i_2\dots<i_g\le 2g-2$, the divisor $P_{i_1}+\dots+P_{i_g}$ is special.

The matrix $\mathcal{B}$ is the Gram matrix of the $2g-2$ vectors from $\mathbb{C}^g$:

$$\mathbf{V_1}=(v_1(P_1),v_2(P_1),\dots,v_g(P_1))$$

....

$$\mathbf{V_{2g-2}}=(v_1(P_{2g-2}),v_2(P_{2g-2}),\dots,v_g(P_{2g-2}))$$

with respect to the (non-degenerate) Hermitian product in $\mathbb{C}^g$,

$$\langle\mathbf{V},\mathbf{W}\rangle=\sum_{\alpha,\beta=1}^{g}\Im\mathbb{B}^{-1}_{\alpha\beta}V_\alpha\bar{W}_\beta\,.$$

Thus, rank $\mathcal{B}\le g$, and if rank $\mathcal{B}=g$, then, according to the Principal Minor Theorem for Hermitian matrices (see, e.g., [1]), there exists a non-zero principal minor of $\mathcal{B}$ of order g. This contradicts (3.1).

(B) $\Rightarrow$ **(A)**: We will use the following elementary fact from linear algebra:

Let E be a vector space of dimension g and let E^* be its dual space. Let $k \geq g$ and $\{L_j\}_{j=1,\dots,k}$ be a collection of linear forms on E. Then

$$\mathrm{Span}_{\,j=1,\dots,k} L_j = E^* \Leftrightarrow \bigcap_{j=1}^{k} \mathrm{Ker}\, L_j = \{0\}$$

$$\Leftrightarrow \exists i_1, \dots, i_g \,:\, L_{i_1}, \dots, L_{i_g} \text{ are linear independent}\,.$$

Let E be the (g-dimensional) space of holomorphic one forms on X, and let $L_j(v) = v(P_j)$, where v is a holomorphic one form and $v(P_j)$ is its value at P_j with respect to the distinguished local parameter of $\mathbf{m}$ at P_j. If $\mathrm{rank}\,\mathcal{B} < g$, then no set of g forms L_j is independent, and, therefore, the intersection of all $\mathrm{Ker}\, L_j$ is not trivial. Thus, there exists a non-zero holomorphic one form vanishing at all P_j. $\square$

Remark 1.[1] The second part of the proof can be simplified when X is non-hyperelliptic by using that in this case $X \ni P \to (v_1(P), \dots, v_g(P)) \in \mathbb{C}P^{g-1}$ is an embedding.

Remark 2. It can be read from the S-matrix whether the Troyanov metric has trivial holonomy or not.

4. Dirichlet-to-Neumann isospectrality and Bergman kernel

4.1. **Comparison formula for determinants and Bergman kernel.** The following proposition is essentially a reformulation of Theorem 1 from [5] for the case of the holomorphic extension Δ_{hol}. Since this reformulation is not completely immediate, we give here some details of the proof.

Proposition 3. *Let*

$$T(\lambda) := S^{ha}(\lambda) = \begin{pmatrix} S^{\xi_1\bar\xi_1}(\lambda) & S^{\xi_1,\bar\xi_2}(\lambda) & \dots & S^{\xi_1,\bar\xi_{2g-2}}(\lambda) \\ \vdots & \vdots & \dots & \vdots \\ S^{\xi_{2g-2}\bar\xi_1}(\lambda) & S^{\xi_{2g-2},\bar\xi_2}(\lambda) & \dots & S^{\xi_{2g-2},\bar\xi_{2g-2}}(\lambda) \end{pmatrix}.$$

There exists a constant C_g that depends only on the genus such that the following comparison formula for the ζ-regularized determinants holds:

$$\det(\Delta_{\mathrm{hol}} - \lambda) = C_g \,\mathrm{detT}(\lambda)\det(\Delta_F - \lambda)\,.$$

To get Proposition 3 one needs the following reformulation of Proposition 5.3 from [5] for the case of the holomorphic extension.

Lemma 1. *The following relation holds true:*

$$\mathrm{Tr}((\Delta_{\mathrm{hol}} - \lambda)^{-1} - (\Delta_F - \lambda)^{-1}) = -\mathrm{Tr}\left(T(\lambda)^{-1}\frac{d}{d\lambda}T(\lambda)\right). \tag{4.1}$$

Proof. As in [5] the key idea is to make use of the M. G. Krein theory for the difference of the resolvents of two self-adjoint extensions of a symmetric operator with finite deficiency indices. Using this theory, for any $f \in L^2(X)$ there exists a collection $(x_k)_{k=1,\dots,2g-2}$ such that

$$(\Delta_{\mathrm{hol}} - \lambda)^{-1} f = (\Delta_F - \lambda)^{-1} f \,+\, \sum_{k=1}^{2g-2} x_k G_{\xi_k}(\cdot, \lambda)\,. \tag{4.2}$$

[1]We thank K. Shramov and N. Tyurin for this remark.

For each j, considering the coefficient of $\bar{\xi}_j$ leads to the following equation:

$$\forall j,\ 0 = \Lambda_{\bar{\xi}_j}\left((\Delta_F-\lambda)^{-1}f\right) + \sum_{k=1}^{2g-2} x_k S^{\xi_k\bar{\xi}_j}.$$

Since $S^{\xi_k\bar{\xi}_j} = T_{kj}$, by inverting this relation we obtain that

$$\forall k,\ x_k = -\sum_{j=1}^{2g-2}\Lambda_{\bar{\xi}_j}\left((\Delta_F-\lambda)^{-1}f\right)T_{jk}^{-1}.$$

It follows that by taking the trace we obtain

(4.3)

$$\begin{aligned}\mathrm{Tr}((\Delta_{\mathrm{hol}}-\lambda)^{-1}-(\Delta_F-\lambda)^{-1}) &= -\sum_{j,k=1}^{2g-2}\sum_{\ell\geq 0}\Lambda_{\bar{\xi}_j}\left((\Delta_F-\lambda)^{-1}\phi_\ell\right)T_{jk}^{-1}\langle\phi_\ell, G_{\xi_k}\rangle\\ &= -\sum_{j,k=1}^{2g-2}T_{jk}^{-1}\sum_{\ell\geq 0}(\lambda_\ell-\lambda)^{-2}\Lambda_{\bar{\xi}_j}(\phi_\ell)\Lambda_{\xi_k}(\phi_\ell).\end{aligned}$$

Using equation (2.4), we get

$$\sum_{\ell\geq 0}(\lambda_\ell-\lambda)^{-2}\Lambda_{\bar{\xi}_j}(\phi_\ell)\Lambda_{\xi_k}(\phi_\ell) = \Lambda_{\bar{\xi}_j}\left(\partial_\lambda G_{\xi_k}\right) = \partial_\lambda S^{\xi_k\bar{\xi}_j} = \partial_\lambda T_{kj}.$$

The lemma thus follows. □

This lemma implies Proposition 3 by performing exactly the same contour integration as in [5] (from that point, the remaining part of the proof can be repeated verbatim).

4.2. **DtN isospectrality of Δ_F and Δ_{hol}.** Let the divisor $P_1+\cdots+P_{2g-2}$ be in the canonical class. Then the corresponding Troyanov metric (all the conical angles are equal to 4π) has trivial holonomy and is given by $|\omega|^2$, where ω is a holomorphic one form with simple zeros at $P_1,\ldots,P_{2g-2}$. Let $z(P)=\int_{P_0}^P\omega$ with some base point P_0. Then $z(P)$ can be taken as holomorphic local parameter on X in a vicinity of any point of $X\setminus\{P_1,\ldots,P_{2g-2}\}$. Consider the operator

$$D_z : L^2(X,|\omega|^2)\supset C_0^\infty(X\setminus\{P_1,\ldots,P_{2g-2}\}\ni u\to u_z\in L^2(X,|\omega|^2).$$

Remark 3. Let ∂_z be the standard Cauchy-Riemann operator $\partial_z : C^\infty(X)\to\Lambda^{1,0}(X)$. Then $D_z=\frac{1}{\omega}\partial_z$.

The operator D_z is closable. Denote its closure again by D_z. The following observation belongs to the first author [4].

Proposition 4. *One has the following identifications of the self-adjoint operators:*

$$D_z^*D_z = \frac{1}{4}\Delta_F,$$

$$D_zD_z^* = \frac{1}{4}\Delta_{\mathrm{hol}}.$$

Proof. By general considerations, both operators are self-adjoint extensions of the Laplace operator on functions that vanish near the conical points. It thus suffices to prove that the domains coincide. For the first operator, it follows by remarking that $\mathrm{dom}(D_z^*D_z) \subset \mathrm{dom}(D_z)$. For the second operator it follows by remarking that any function in $\mathrm{dom}(\Delta^*)$ with holomorphic behaviour near the conical points is in $\mathrm{dom}(D_zD_z^*)$. □

From Proposition 4 it immediately follows that

$$\forall\lambda \neq 0,\ \dim\mathrm{Ker}\,(\Delta_F - \lambda) = \dim\mathrm{Ker}\,(\Delta_{\mathrm{hol}} - \lambda). \tag{4.4}$$

Definition 2. We will say that Δ_F and Δ_{hol} are DtN-isospectral to indicate that (4.4) holds true.

To compare the spectral determinants of both extensions, it thus suffices to compute the dimension of their kernels. The kernel of Δ_F only consists of the constant functions.

Lemma 2. *For metrics with trivial holonomy one has the equality*

$$\dim\mathrm{Ker}\,\Delta_{\mathrm{hol}} = g\,.$$

Proof. Denote by X_ϵ the surface X where ϵ-disks around P_k have been deleted. Then, for any $u \in \mathrm{dom}(\Delta_{\mathrm{hol}})$ we have

$$\begin{aligned}
-\frac{1}{4}\langle u, \Delta u\rangle &= \lim_{\epsilon\to 0}\int_{X_\epsilon} \bar{u}\partial_z\partial_{\bar z}u|dz|^2\\
&= \langle \partial_{\bar z}u, \partial_{\bar z}u\rangle\\
&+ \lim_{\epsilon\to 0}\sum_{k=1}^{2g-2}\oint_{|\xi_k|=\sqrt{\epsilon}}(A_k/\xi_k + \dots)\frac{1}{\bar{\xi}_k}\partial_{\bar{\xi}_k}(A_k/\xi_k + B + C\xi_k + O(|\xi_k|^2))\bar{\xi}_k d\bar{\xi}_k\\
&= \langle \partial_{\bar z}u, \partial_{\bar z}u\rangle.
\end{aligned}$$

It follows that if $u \in \mathrm{Ker}\,\Delta_{\mathrm{hol}}$, then $\partial_{\bar z}u = 0$. Thus, the one form $u\omega$ is holomorphic and, therefore, $u \in \mathrm{Span}\{v_1/\omega, \dots, v_g/\omega\}$. Conversely, for any holomorphic one form v the meromorphic function v/ω is seen to be in $\mathrm{dom}(\Delta_{\mathrm{hol}})$ and satisfies $\Delta^* v/\omega = 0$. We obtain

$$\ker(\Delta_{\mathrm{hol}}) \;=\; \mathrm{Span}\{v_1/\omega, \dots, v_g/\omega\}.$$

□

Remark 4. The needed equality also follows from the DtN isospectrality of Δ_{hol} and Δ_F and Remark 5.11 from [5]. Indeed, the latter implies the relation

$$\zeta(0, \Delta_{hol} - \lambda) = \zeta(0, \Delta_F - \lambda) + g - 1$$

between the values of the operator zeta-functions of $\Delta_{\mathrm{hol}} - \lambda$ and $\Delta_F - \lambda$ at $s = 0$.

The DtN isospectrality of Δ_{hol} and Δ_F (see [4], Theorem 4.8) then implies that

$$\det(\Delta_{\mathrm{hol}} - \lambda) = \lambda^{g-1}\det(\Delta_F - \lambda)$$

and, therefore,

$$\frac{\det T(\lambda)}{\lambda^{g-1}} = C_g \tag{4.5}$$

and

$$C_g = \frac{1}{(g-1)!}\left(\frac{d}{d\lambda}\right)^{g-1}\det T(\lambda)\Big|_{\lambda=0}. \tag{4.6}$$

For any integer M, denote by $\mathcal{N}_M$ the set of $(2g-2)$-tuples $\mathbf{n} := (n_1, \dots, n_{2g-2})$ of non-negative integers such that $n_1 + \cdots + n_{2g-2} = M$. The preceding identity can thus be reformulated as

(4.7)
$$C_g = \sum_{\mathbf{n}\in\mathcal{N}(g-1)} \frac{1}{n_1!\dots n_{2g-2}!}\begin{vmatrix} \left(\frac{d}{d\lambda}\right)^{n_1} S^{\xi_1\bar{\xi}_1}(0) & \cdots & \left(\frac{d}{d\lambda}\right)^{n_{2g-2}} S^{\xi_1\bar{\xi}_{2g-2}}(0) \\ \vdots & \cdots & \vdots \\ \left(\frac{d}{d\lambda}\right)^{n_1} S^{\xi_{2g-2}\bar{\xi}_1}(0) & \cdots & \left(\frac{d}{d\lambda}\right)^{n_{2g-2}} S^{\xi_{2g-2}\bar{\xi}_{2g-2}}(0)\end{vmatrix}$$

and C_g is an absolute constant that depends on the genus only.

We now give alternative expressions for the coefficients in the preceding determinant.

Lemma 3. *For all $n \geq 1$ and all j, k, the following generalization of the formula* (4.2) *from* [6] *holds:*

$$(\partial_\lambda)^n S^{\xi_k\bar{\xi}_j}(\lambda) = n!\int_X [(\Delta_F - \lambda)^{1-n} G_{\xi_k}]G_{\bar{\xi}_j}\, dS. \tag{4.8}$$

Proof. We start from (2.4), apply $\Lambda_{\bar{\xi}_j}$ and differentiate $n-1$ times to obtain

$$\begin{aligned}(\partial_\lambda)^n S^{\xi_k\bar{\xi}_j}(\lambda) &= n!\sum_{\ell\geq 0}(\lambda_\ell - \lambda)^{-1-n}\Lambda_{\xi_k}(\phi_\ell)\Lambda_{\bar{\xi}_j}(\phi_\ell) \\ &= n!\sum_{\ell\geq 0}(\lambda_\ell-\lambda)^{1-n}\frac{\Lambda_{\xi_k}(\phi_\ell)}{\lambda_\ell-\lambda}\frac{\Lambda_{\bar{\xi}_j}(\phi_\ell)}{\lambda_\ell-\lambda} \\ &= n!\int_X [(\Delta_F-\lambda)^{1-n}G_{\xi_k}]G_{\bar{\xi}_j}\, dS.\end{aligned}$$

□

From the expansion in the eigenfunctions basis, we now observe that $G_{\xi_k}\perp 1$ and is analytic at 0. It follows that

$$G_{\xi_k}(\cdot;\lambda=0) = \mathcal{H}_k := \frac{1}{2\sqrt{\pi}}\big(H_k - \frac{1}{A}\int_X H_k\big), \tag{4.9}$$

where H_k is explicitly given by (2.8).

The operator

$$\Delta_F^{-1} : 1^\perp \to 1^\perp$$

has an integral kernel which is given by the Green function

$$G(x,y) = \frac{1}{2\pi A^2}\int_X\int_X \Re\left(\int_b^x \Omega_{y-a}\right) dS(a)dS(b) \tag{4.10}$$

where

$$\Omega_{a-b}(z) = \int_a^b W(z,\cdot) - 2\pi i\sum_{\alpha\beta}(\Im\mathbb{B})^{-1}_{\alpha\beta}v_\alpha(z)\Im\int_a^b v_\beta$$

(see [3], formula (2.19)). Thus, by letting λ go to 0 we obtain the following explicit expression for all the terms in the r.h.s. of (4.7):

$$\left(\frac{d}{d\lambda}\right)^{n_k}_{|\lambda=0} S^{\xi_l\bar{\xi}_k}(\lambda) = \int_X [(\Delta_F|_{1^\perp})^{1-n_k}\mathcal{H}_l]\overline{\mathcal{H}_k} \tag{4.11}$$

where

$$(\Delta_F|_{1^\perp})^{1-n_k}\mathcal{H}_l(x) \tag{4.12}$$

$$= \int_X G(x,x_1)\int_X G(x_1,x_2)\cdots\int_X G(x_{n_k-1},y)\mathcal{H}_l(y)dS(y)dS(x_{n_k-1})\dots dS(x_1)\,.$$

5. Metrics with non-trivial holonomy

Lemma 4. *Let $P_1,\dots,P_{2g-2}$ be points of X such that the divisor $D = P_1+\dots+P_{2g-2}$ is not canonical. (According to Proposition 2, this condition is equivalent to the condition $\operatorname{rank}\mathcal{B} = g$.) Then we have*

$$\dim\operatorname{Ker}\Delta_{\text{hol}} = g-1\,.$$

Proof. Let $u\in\operatorname{Ker}\Delta_{\text{hol}}$. Then, similarly to the proof of Lemma 1, one can show that u is holomorphic in $X\setminus\{P_1,\dots P_{2g-2}\}$. According to the corollary of Theorem III.9.10 from [2], D is the divisor of a holomorphic multi-valued Prym differential ω with non-trivial (multiplicative) unitary monodromy T_{a_α}, T_{b_α} along basic cycles a_α, b_α:

$$T_{a_\alpha}\omega = e^{ir_\alpha}\omega; T_{b_\alpha}\omega = e^{is_\alpha}\omega;\quad r_\alpha, s_\alpha\in\mathbb{R}\,.$$

Then $u\omega$ is a Prym differential with the same monodromy. Since the space of Prym differentials with a given non-trivial unitary monodromy has dimension $g-1$ (see [10], p. 147), one gets

$$\text{dimker}\Delta_{\text{hol}}\le g-1\,.$$

The inequality

$$\text{dimker}\Delta_{\text{hol}}\ge g-1$$

is obvious: take $\{\omega_1,\dots,\omega_{g-1}\}$ as a basis of the space of Prym differentials with the same monodromy as ω. Then the functions $\omega_1/\omega,\dots,\omega_{g-1}/\omega$ are linearly independent and belong to $\text{dom}(\Delta_{\text{hol}})\cap\ker\Delta^*$. □

Lemma 4 and Proposition 3 imply the following theorem.

Theorem 2. *Let the divisor $P_1+\dots+P_{2g-2}$ not belong to the canonical class. Then*

$$\frac{\det\Delta_{\text{hol}}}{\det\Delta_F}$$

$$= c_g\cdot\sum_{\mathbf{n}\in\mathcal{N}(g-2)}\frac{1}{n_1!\dots n_{2g-2}!}\begin{vmatrix}\left(\frac{d}{d\lambda}\right)^{n_1}S^{\xi_1\bar{\xi}_1}(0) & \dots & \left(\frac{d}{d\lambda}\right)^{n_{2g-2}}S^{\xi_1\bar{\xi}_{2g-2}}(0)\\ \vdots & \dots & \vdots\\ \left(\frac{d}{d\lambda}\right)^{n_1}S^{\xi_{2g-2}\bar{\xi}_1}(0) & \dots & \left(\frac{d}{d\lambda}\right)^{n_{2g-2}}S^{\xi_{2g-2}\bar{\xi}_{2g-2}}(0)\end{vmatrix},$$

where all the terms at the right are explicitly given by (4.9, 4.10, 4.11, 4.12) *and the absolute constant c_g depends only on the genus g. In particular, in genus* 2 *one gets relation* (1.3).

Acknowledgements

The authors thank D. Korotkin for numerous discussions. They are extremely grateful to K. Shramov and N. Tyurin for their help with the proof of Proposition 2; the second author also acknowledges useful discussions with P. Zograf, in particular, the question about the relation between det Δ_F and det $\Delta_{\rm hol}$ was raised by P. Zograf in one of these discussions.

The research of the second author was supported by NSERC.

References

[1] M. F. Egan, *Symmetric matrices and quadratic forms*, Math. Gaz. **29** (1945), 89–91. MR0013123

[2] Hershel M. Farkas and Irwin Kra, *Riemann surfaces*, Graduate Texts in Mathematics, vol. 71, Springer-Verlag, New York-Berlin, 1980. MR583745

[3] John Fay, *Kernel functions, analytic torsion, and moduli spaces*, Mem. Amer. Math. Soc. **96** (1992), no. 464, vi+123, DOI 10.1090/memo/0464. MR1146600

[4] Luc Hillairet, *Spectral theory of translation surfaces: a short introduction* (English, with English and French summaries), Actes du Séminaire de Théorie Spectrale et Géometrie. Volume 28. Année 2009–2010, Sémin. Théor. Spectr. Géom., vol. 28, Univ. Grenoble I, Saint-Martin-d'Hères, 201?, pp. 51–62, DOI 10.5802/tsg.278. MR2848211

[5] Luc Hillairet and Alexey Kokotov, *Krein formula and S-matrix for Euclidean surfaces with conical singularities*, J. Geom. Anal. **23** (2013), no. 3, 1498–1529, DOI 10.1007/s12220-012-9295-3. MR3078362

[6] L. Hillairet, V. Kalvin, and A. Kokotov, Moduli spaces of meromorphic functions and determinant of Laplacian, arXiv:1410.3106 [math.SP]

[7] Alexey Kokotov, *Polyhedral surfaces and determinant of Laplacian*, Proc. Amer. Math. Soc. **141** (2013), no. 2, 725–735, DOI 10.1090/S0002-9939-2012-11531-X. MR2996977

[8] Aleksey Kokotov and Dmitry Korotkin, *Tau-functions on spaces of abelian differentials and higher genus generalizations of Ray-Singer formula*, J. Differential Geom. **82** (2009), no. 1, 35–100. MR2504770

[9] Marc Troyanov, *Les surfaces euclidiennes à singularités coniques* (French), Enseign. Math. (2) **32** (1986), no. 1-2, 79–94. MR850552

[10] Hermann Weyl, *The concept of a Riemann surface*, translated from the third German edition by Gerald R. MacLane. ADIWES International Series in Mathematics, Addison-Wesley Publishing Co., Inc., Reading, Mass.-London, 1964. MR0166351

MAPMO (UMR 7349 Université d'Orléans-CNRS) UFR Sciences, Bâtiment de mathématiques rue de Chartres, BP 6759 45067 Orléans Cedex 02, France

E-mail address: luc.hillairet@univ-orleans.fr

Department of Mathematics and Statistics, Concordia University, 1455 de Maisonneuve Boulevard West, Montreal, Quebec, H3G 1M8 Canada

E-mail address: alexey.kokotov@concordia.ca