On G-Function of Frobenius Manifolds Related to Hurwitz Spaces

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1 Introduction

We deal with the class of Frobenius manifolds related to Hurwitz spaces of moduli of meromorphic functions on Riemann surfaces (see [3]).

The key observation in the present paper is the identification of the isomonodromic tau-function (see [3, 4, 5, 6]) of this class of Frobenius manifolds with $(-1/2)$-power of the Bergmann tau-function which was introduced in [10] in a rather different context. We show that the quadratic Hamiltonian from [3] coincides (up to a constant) with the value of the Bergmann projective connection calculated in the natural local parameter at the critical point of the meromorphic function. This simple observation enables us to apply the results of [10] and explicitly calculate the isomonodromic tau-functions of Frobenius manifolds related to the Hurwitz spaces of moduli of meromorphic functions on surfaces of genus 0 and 1. This immediately leads to general formulas for the G-function (see [5, 6]) of the above Frobenius manifolds. We recall that the G-function of a Frobenius manifold provides a solution of the so-called Getzler equation (see [6, 8]). For some classes of Frobenius manifolds, it plays the role of a generating function of Gromov-Witten invariants of algebraic varieties (see [5]). In the general case, it gives the genus one free energy of Dijkgraaf and Witten and describes first-order deformations of dispersionless integrable systems.

As a consequence of our general result, we prove the following formula for the G-function of the Frobenius manifold $\mathbb{C} \times \mathbb{C}^{N-1} \times \{\Im z > 0\}/J(A_{N-1})$ which was recently

conjectured by Strachan [14]:

$$G = -\ln \eta(t_0) - \frac{N+1}{24} t_N. \tag{1.1}$$

Moreover, using the results of [10], we get the expression for the modulus square of the isomonodromic tau-function (and, hence, for the real part of the G-function) in case of Hurwitz spaces in higher genus.

2   Preliminaries

In this section, we briefly outline some basic facts and definitions from the theory of Frobenius manifolds (see [1, 2, 3, 4, 5, 6, 11, 12]).

2.1   Hurwitz spaces and Frobenius manifolds

Here we mainly follow [3, Lecture 5], departing somewhat from Dubrovin's original notation. Let $H_{g,N}(k_1, \ldots, k_l)$ be the Hurwitz space of equivalence classes $[p : \mathcal{L} \to \mathbb{P}^1]$ of N-fold branched coverings

$$p : \mathcal{L} \longrightarrow \mathbb{P}^1, \tag{2.1}$$

where $\mathcal{L}$ is a compact Riemann surface of genus g and the holomorphic map p of degree N is subject to the following conditions:

(i)   it has M simple ramification points $P_1, \ldots, P_M \in \mathcal{L}$ with distinct *finite* images $\lambda_1, \ldots, \lambda_M \in \mathbb{C} \subset \mathbb{P}^1$;

(ii) the preimage $p^{-1}(\infty)$ consists of l points: $p^{-1}(\infty) = \{\infty_1, \ldots, \infty_l\}$, and the ramification index of the map p at the point ∞_j is k_j ($1 \leq k_j \leq N$).

(The ramification index at a point is equal to the number of sheets of the covering which are glued at this point. A point ∞_j is a ramification point if and only if $k_j > 1$. A ramification point is simple if the corresponding ramification index equals 2.)

Notice that $k_1 + \cdots + k_l = N$ and $M = 2g + l + N - 2$. (The last equality is a consequence of the Riemann-Hurwitz formula.) Two branched coverings $p_1 : \mathcal{L}_1 \to \mathbb{P}^1$ and $p_2 : \mathcal{L}_2 \to \mathbb{P}^1$ are called equivalent if there exists a biholomorphic map $f : \mathcal{L}_1 \to \mathcal{L}_2$ such that $p_2 f = p_1$.

The Hurwitz spaces $H_{g,N}(k_1, \ldots, k_l)$ can also be described as spaces of meromorphic functions of degree N on Riemann surfaces of genus g with l poles of orders $k_1, \ldots, k_l$ and simple critical values.

For example, the space $H_{0,N}(N)$ has an equivalent description as the space of polynomials $\mathcal{L} = \mathbb{P}^1 \ni z \mapsto \lambda(z) \in \mathbb{P}^1$,

$$\lambda(z) = z^N + a_2 z^{N-2} + a_3 z^{N-3} + \cdots + a_N, \tag{2.2}$$

whereas the space $H_{0,N}(k, N-k)$ $(1 \leq k \leq N-1)$ can be described as the space of "trigonometric polynomials" $\mathcal{L} = \mathbb{P}^1 \ni z \mapsto \lambda(z) \in \mathbb{P}^1$,

$$\lambda(z) = z^k + b_1 z^{k-1} + \cdots + \frac{b_N}{z^{N-k}}, \quad b_N \neq 0. \tag{2.3}$$

We assume that the critical values of $\lambda(z)$ in (2.2) and (2.3) are simple (i.e., the derivative $\lambda'(z)$ has only simple roots and $\lambda(z_i) \neq \lambda(z_j)$ for distinct roots z_1 and z_2 of $\lambda'(z)$).

We also introduce the covering $\widehat{H}_{g,N}(k_1, \ldots, k_l)$ of the space $H_{g,N}(k_1, \ldots, k_l)$ consisting of pairs

$$\left\langle \left[p : \mathcal{L} \longrightarrow \mathbb{P}^1 \right] \in H_{g,N}(k_1, \ldots, k_l), \ \{a_\alpha, b_\alpha\}_{\alpha=1}^g \right\rangle, \tag{2.4}$$

where $\{a_\alpha, b_\alpha\}_{\alpha=1}^g$ is a canonical basis of cycles on the Riemann surface $\mathcal{L}$.

Obviously, for $g = 0$, the spaces $H_{0,N}(k_1, \ldots, k_l)$ and $\widehat{H}_{0,N}(k_1, \ldots, k_l)$ coincide.

The spaces $H_{g,N}(k_1, \ldots, k_l)$ and $\widehat{H}_{g,N}(k_1, \ldots, k_l)$ are connected complex manifolds of dimension $M = 2g + l + N - 2$ and the local coordinates on these manifolds are given by the finite critical values of the map p (or, equivalently, the finite branch points of the covering (2.1)) $\lambda_1, \ldots, \lambda_M$.

In [3], the notion of the so-called "primary" differentials on the Riemann surfaces $\mathcal{L}$ was introduced; each primary differential ϕ defines a structure of Frobenius manifold M_ϕ on $\widehat{H}_{g,N}(k_1, \ldots, k_l)$. We will not reproduce here the complete list of primary differentials (see [3]). We only notice that in the case $g \geq 1$, the normalized $(\int_{a_\alpha} \omega_\beta = \delta_{\alpha\beta})$ holomorphic differentials ω_β on Riemann surfaces $\mathcal{L}$ are primary differentials. The (meromorphic) differentials dz and dz/z on the Riemann sphere $\mathcal{L}$ are primary differentials in cases of the spaces $H_{0,N}(N)$ and $H_{0,N}(k, N-k)$, respectively.

The structure of Frobenius manifold M_ϕ on $\widehat{H}_{g,N}(k_1, \ldots, k_l)$ is defined by the multiplication law in the tangent bundle: $\partial_{\lambda_m} \circ \partial_{\lambda_n} = \delta_{mn} \partial_{\lambda_m}$, the unity $e = \sum_{m=1}^M \partial_{\lambda_m}$, the Euler field $E = \sum_{m=1}^M \lambda_m \partial_{\lambda_m}$, and the one-form $\Omega_{\phi^2} = \sum_{m=1}^M \{\mathrm{Res}_{P_m}(\phi^2/d\lambda)\} d\lambda_m$,

where λ is the coordinate on the $\mathcal{L}$ lifted from the base $\mathbb{P}^1$. The invariant metric $\eta(v, w) = \Omega_{\phi^2}(v \circ w)$ on the Frobenius manifold turns out to be flat and potential (i.e., Egoroff-Darboux metric). In the coordinates $\lambda_1, \dots, \lambda_M$ (which are called *canonical*), this metric is diagonal:

$$\eta = \sum_{m=1}^{M} \eta_{mm} (d\lambda_m)^2, \quad \eta_{mm} = \operatorname{Res}_{P_m} \left(\frac{\phi^2}{d\lambda} \right) \tag{2.5}$$

and its rotation coefficients $\gamma_{mn} = \partial_{\lambda_n} \sqrt{\eta_{mm}}/\sqrt{\eta_{nn}}$ (where $m \neq n$) have the following properties. First, they are independent of the choice of a primary differential ϕ. Second, they satisfy the equations

$$\partial_{\lambda_k} \gamma_{mn} = \gamma_{mk} \gamma_{kn}, \quad \text{for distinct } k, n, m, \tag{2.6}$$

$$e(\gamma_{mn}) = \sum_{k=1}^{M} \partial_{\lambda_k} \gamma_{mn} = 0, \tag{2.7}$$

which provide the flatness of metric (2.5). Finally, the action of the Euler vector field on γ_{mn} has the form

$$E(\gamma_{mn}) = \sum_{k=1}^{M} \lambda_k \partial_{\lambda_k} \gamma_{mn} = -\gamma_{mn}. \tag{2.8}$$

The following three examples of Frobenius manifolds related to Hurwitz spaces are of special interest since they also arise in the theory of Coxeter, extended affine Weyl, and Jacobi groups, respectively (see [1, 2, 3, 4]).

(i) $M_{0;N}$. The underlying Hurwitz space here is the space $H_{0,N}(N)$. In this case $g = 0, l = 1$, and the primary differential defining the structure of Frobenius manifold is dz.

(ii) $M_{0;k,N-k}$. The underlying Hurwitz space is $H_{0,N}(k, N-k)$, $g = 0, l = 2$, $1 \leq k \leq N-1$, and the Frobenius structure is defined by the primary differential dz/z.

(iii) $\widehat{M}_{1,N}$. The underlying space here is the covering $\widehat{H}_{1,N}(N)$, $g = 1, l = 1$, and the primary differential on the elliptic surface $\mathcal{L}$ is the normalized ($\int_a \omega = 1$) holomorphic differential ω.

Due to [4], the first $N - 1$ flat coordinates of the metric η in case of the Frobenius manifold $M_{0;k,N-k}$ of dimension $M = N$ are given by

$$t_\mu = (-1)^{\mu+1}\frac{k}{\mu}\operatorname*{Res}_{z=\infty}\left[\lambda(z)\right]^{\mu/k}d\ln z, \quad 1 \leq \mu < k-1,$$

$$t_{N-\mu} = (-1)^{\mu}\frac{N-k}{\mu}\operatorname*{Res}_{z=0}\left[(-1)^N\lambda(z)\right]^{\mu/(N-k)}d\ln z, \quad 1 \leq \mu \leq N-k. \tag{2.9}$$

The last flat coordinate t_N is defined by the equation

$$b_N = (-1)^N \exp\left[(N-k)t_N\right]. \tag{2.10}$$

To write down the flat coordinates on the Frobenius manifold $\widehat{M}_{1;N}$ (of dimension $N+1$), set $z(P) = \int_{\infty_1}^P \omega$, where ∞_1 is the point on $\mathcal{L}$ such that $p(\infty_1) = \infty$ and $\lambda(z(P)) = p(P)$. Then the flat coordinates $t_0, \ldots, t_N$ are given by (see [1, 2]) $t_0 = \int_b \omega = \sigma$, where σ is the modulus of the elliptic curve $\mathcal{L}$, $t_1 = \int_a \lambda(z(P))dz(P)$, and

$$t_\mu = \operatorname*{Res}_{z=0} z\left[\lambda(z)\right]^{-(\mu-1)/N}d\lambda(z), \quad \mu = 2, \ldots, N. \tag{2.11}$$

2.2 Isomonodromic tau-function and G-function of Frobenius manifold

Let M_ϕ be the Frobenius manifold with an underlying Hurwitz space $\widehat{H}_{g,N}(k_1, \ldots, k_l)$ and whose Frobenius structure is given by a primary differential ϕ. Set $\Gamma = \|\gamma_{mn}\|_{m,n=1,\ldots,M}$ (the diagonal elements of the matrix Γ are not defined), $U = \operatorname{diag}(\lambda_1, \ldots, \lambda_M)$, and $V = [\Gamma, U]$. Here γ_{mn} are the rotation coefficients of the metric (2.5), and $\lambda_1, \ldots, \lambda_M$ are the canonical coordinates on M_ϕ. The matrix V is well defined since the diagonal elements of Γ do not enter the commutator $[\Gamma, U]$.

The isomonodromic tau-function τ_I of the Frobenius manifold M_ϕ is defined by the system of (compatible) equations

$$\frac{\partial \ln \tau_I}{\partial \lambda_m} = H_m, \quad m = 1, \ldots, M, \tag{2.12}$$

where the Hamiltonians H_m are defined by

$$H_m = \frac{1}{2}\sum_{n \neq m; \, 1 \leq n \leq M}\frac{V_{nm}^2}{\lambda_m - \lambda_n}, \quad m = 1, \ldots, M. \tag{2.13}$$

Let $t_1, \ldots, t_M$ be the flat coordinates on the Frobenius manifold M_ϕ. The Jacobian $J = \det\|\partial\lambda_m/\partial t_n\|$ can be expressed in terms of metric coefficients η_{mm} as follows:

$$J = \left(\prod_{m=1}^M \eta_{mm}\right)^{1/2} = \left(\prod_{m=1}^M \operatorname{Res}_{P_m}\frac{\phi^2}{d\lambda}\right)^{1/2}. \tag{2.14}$$

The G-function of the Frobenius manifold M_ϕ is defined as:

$$G = \ln\left(\frac{\tau_I}{J^{1/24}}\right). \tag{2.15}$$

3   Isomonodromic tau-function of Frobenius manifold and Bergmann tau-function on Hurwitz space

3.1   Rotation coefficients of the flat metric η and the Bergmann kernel

First, we recall the definition of the Bergmann kernel. In the case $g > 0$, the Bergmann kernel on the Torelli marked Riemann surface $\mathcal{L}$ is defined by $B(P, Q) = d_P d_Q \ln E(P, Q)$, where $E(P, Q)$ is the prime form on $\mathcal{L}$ (see [7]). At the diagonal $P = Q$, the Bergmann kernel is singular:

$$B\big(x(P), x(Q)\big) = \left(\frac{1}{\big(x(P) - x(Q)\big)^2} + H\big(x(P), x(Q)\big)\right) dx(P)\, dx(Q), \tag{3.1}$$

where

$$H\big(x(P), x(Q)\big) = \frac{1}{6} S_B\big(x(P)\big) + o(1) \tag{3.2}$$

as $P \to Q$. Here $x(P)$ is a local coordinate of a point $P \in \mathcal{L}$ and S_B is the Bergmann projective connection (see, e.g., [7, 15]).

If $g = 0$ and $z : \mathcal{L} \to \mathbb{P}^1$ is a biholomorphic map, then the Bergmann kernel is defined by

$$B\big(z(P), z(Q)\big) = \frac{dz(P)dz(Q)}{\big(z(P) - z(Q)\big)^2}. \tag{3.3}$$

(In particular, $S_B(z) \equiv 0$ in the local parameter z.)

Near a simple ramification point $P_m \in \mathcal{L}$ of covering (2.1), we introduce the local parameter

$$x_m(P) = \big(\lambda(P) - \lambda_m\big)^{1/2}, \tag{3.4}$$

where $\lambda(P) = p(P), \lambda_m = p(P_m), m = 1, \dots, M$.

Let $U(P_m)$ and $U(P_n)$ be small neighborhoods of ramification points P_m and P_n, respectively. For $(P, Q) \in U(P_m) \times U(P_n)$, we set

$$b_{mn}(P, Q) = \frac{B\big(x_m(P), x_n(Q)\big)}{dx_m(P)\, dx_n(Q)}. \tag{3.5}$$

Lemma 3.1 (cf. [9])**.** The rotation coefficients γ_{mn} of the metric

$$\eta = \sum_{m=1}^{M} \operatorname{Res}_{P_m} \left(\frac{\phi^2}{d\lambda}\right)(d\lambda_m)^2 \tag{3.6}$$

are related to $b_{mn}(P, Q)$ as follows:

$$\gamma_{mn} = \frac{1}{2} b_{mn}\big(P_m, P_n\big), \quad m, n = 1, \dots, M, \ m \neq n. \tag{3.7}$$
$$\square$$

Proof. For $g \geq 1$, the proof is in [9]. In brief, it goes as follows. Since the rotation coefficients are independent of the choice of a primary differential ϕ, it is sufficient to verify (3.7) only in the case $\phi = \omega_1$, where ω_1 is the holomorphic differential on $\mathcal{L}$ such that $\int_{a_\alpha} \omega_1 = \delta_{1\alpha}$. For such a primary differential, we have

$$\eta_{mm} = \operatorname{Res}_{P_m} \frac{\omega_1^2}{d\lambda} = \frac{1}{2} \left[\left. \frac{\omega_1\big(x_m(P)\big)}{dx_m(P)} \right|_{P=P_m} \right]^2. \tag{3.8}$$

Now, (3.7) follows from the definition of rotation coefficients and the Rauch formula:

$$\frac{\partial}{\partial \lambda_n} \left[\left. \frac{\omega_1\big(x_m(P)\big)}{dx_m(P)} \right|_{P=P_m} \right] = \frac{1}{2} b_{mn}\big(P_m, P_n\big) \left[\left. \frac{\omega_1\big(x_n(P)\big)}{dx_n(P)} \right|_{P=P_n} \right]. \tag{3.9}$$

Consider the case $g = 0$. Let $z : \mathcal{L} \to \mathbb{P}^1$ be a biholomorphic map such that $z(\infty_1) = \infty$. Then $\phi = dz$ is a primary differential in the sense of Dubrovin. For this primary differential,

$$\eta_{mm} = \operatorname*{Res}_{x_m=0} \frac{\big[z'(x_m)\, dx_m\big]^2}{2x_m\, dx_m} = \frac{1}{2} \Big\{ z'(x_m)\big|_{x_m=0} \Big\}^2. \tag{3.10}$$

We prove an analog of Rauch's variational formula for the meromorphic differential dz. Setting $\alpha_m = z'(x_m)|_{x_m=0}$, we get

$$\begin{aligned}
\frac{\partial}{\partial \lambda_n}\{dz\} &= \frac{\partial}{\partial \lambda_n} \left[\big(\alpha_m + O\big(\sqrt{\lambda - \lambda_m}\big)\big) \frac{d\lambda}{2\sqrt{\lambda - \lambda_m}} \right] \\
&= \left(\frac{\delta_{mn}\alpha_m}{2x_m^2} + O(1) \right) dx_m
\end{aligned} \tag{3.11}$$

as $x_m \to 0$. Thus, the meromorphic differential $(\partial/\partial\lambda_n)dz$ has the only pole at P_n and, therefore,

$$\frac{\partial}{\partial\lambda_n}\{dz(P)\} = \frac{1}{2}\left[\left.\frac{B(P, x_n)z'(x_n)}{dx_n}\right|_{x_n=0}\right]. \tag{3.12}$$

On the other hand, as $P \to P_m$ for $m \neq n$, we have

$$\begin{aligned}
\frac{\partial}{\partial\lambda_n}\{dz\} &= \frac{\partial}{\partial\lambda_n}\big(\alpha_m + O(x_m)\big)dx_m \\
&= \left(\frac{\partial\alpha_m}{\partial\lambda_n} + O(x_m)\right)dx_m.
\end{aligned} \tag{3.13}$$

Thus, due to (3.12), we get the following analog of the Rauch formula (3.9):

$$\frac{\partial\alpha_m}{\partial\lambda_n} = \left.\frac{\frac{\partial}{\partial\lambda_n}dz(x_m)}{dx_m}\right|_{x_m=0} = \frac{1}{2}b_{mn}(P_m, P_n)\alpha_n. \tag{3.14}$$

Now, (3.7) follows from (3.10), (3.14), and the definition of rotation coefficients.
∎

Remark 3.2. Lemma 3.1 clarifies properties (2.6), (2.7), and (2.8) of the rotation coefficients. Namely, property (2.6) is nothing but the Rauch variational formula for the Bergmann kernel, and (2.7) and (2.8) follow from the invariance of the Bergmann kernel under the translations $\lambda \mapsto \lambda + \epsilon$ and (respectively) the dilatations $\lambda \mapsto (1+\delta)\lambda$ of every sheet of the covering (2.1).

3.2   The Bergmann tau-function

Introduce the quantities

$$\mathcal{B}_m = -\frac{1}{12}S_B(x_m)\Big|_{x_m=0}, \quad m = 1,\ldots,M, \tag{3.15}$$

where S_B is the Bergmann projective connection from (3.2), x_m is, as usual, the local parameter (3.4) near the ramification point P_m. In [10], we introduced the so-called Bergmann tau-function τ_B on the Hurwitz space $\widehat{H}_{g,N}$ which is defined by the system of equations

$$\frac{\partial\ln\tau_B}{\partial\lambda_m} = \mathcal{B}_m, \quad m = 1,\ldots,M. \tag{3.16}$$

The local solvability of system (3.16) can be obtained, in particular, from the symmetry of the Bergmann kernel and the first statement of the following lemma.

Lemma 3.3. The quantities $\mathcal{B}_m$ satisfy the following equations:

$$\partial_{\lambda_n} \mathcal{B}_m = -\frac{1}{4} b_{mn}^2 (P_m, P_n), \quad m \neq n, \tag{3.17}$$

$$e(\mathcal{B}_m) = \sum_{n=1}^{M} \partial_{\lambda_n} \mathcal{B}_m = 0, \tag{3.18}$$

$$E(\mathcal{B}_m) = \sum_{n=1}^{M} \lambda_n \partial_{\lambda_n} \mathcal{B}_m = -\mathcal{B}_m. \tag{3.19}$$

$\square$

Proof. Since the singular part of the Bergmann kernel in a neighborhood of the ramification point P_m is independent of $\{\lambda_n\}$, we have

$$\partial_{\lambda_n} \mathcal{B}_m = -\frac{1}{2} \left\{ \partial_{\lambda_n} b_{mm}(P, Q) \right\} \Big|_{P=Q=P_m}. \tag{3.20}$$

Computing the right-hand side of (3.20) via the Rauch formula for the Bergmann kernel

$$\partial_{\lambda_m} b_{nk}(P, Q) = \frac{1}{2} b_{nm}(P, P_m) b_{mk}(P_m, Q), \tag{3.21}$$

we get (3.17).

Under the translation $\lambda \mapsto \lambda + \epsilon$ and the dilatation $\lambda \mapsto (1+\delta)\lambda$ of each sheet of covering (2.1) (both transformations generate conformal isomorphisms of $\mathcal{L}$), the Bergmann kernel remains invariant:

$$B^\epsilon (P^\epsilon, Q^\epsilon) = B^\delta (P^\delta, Q^\delta) = B(P, Q). \tag{3.22}$$

We have the following transformation rules for the local parameter x_m and the critical values λ_m:

$$\begin{aligned}
x_m^\epsilon (P^\epsilon) = x_m(P), \qquad & x_m^\delta (P^\delta) = (1+\delta)^{1/2} x_m(P), \\
\lambda_m^\epsilon = \lambda_m + \epsilon, \qquad & \lambda_m^\delta = (1+\delta)\lambda_m.
\end{aligned} \tag{3.23}$$

Therefore, the function H from (3.1) transforms as follows:

$$H^\epsilon \left(x_m^\epsilon (P^\epsilon), x_m^\epsilon (Q^\epsilon) \right) = H \left(x_m(P), x_m(Q) \right), \tag{3.24}$$

$$H^\delta \left(x_m^\delta (P^\delta), x_m^\delta (Q^\delta) \right) = \frac{1}{1+\delta} H \left(x_m(P), x_m(Q) \right). \tag{3.25}$$

Differentiating (3.24) and (3.25) with respect to ϵ and δ, respectively, we get

$$\frac{dH^\epsilon}{d\epsilon} = \sum_n \frac{\partial H^\epsilon}{\partial \lambda_n^\epsilon} = 0, \tag{3.26}$$

$$\frac{dH^\delta}{d\delta} = \sum_n \lambda_n \frac{\partial H^\delta}{\partial \lambda_n^\delta} = -\frac{1}{(1+\delta)^2} H. \tag{3.27}$$

Setting $\epsilon = 0$ and $\delta = 0$, and then $P = Q$ in (3.26) and (3.27), we get (3.18) and (3.19). ∎

3.3   The relation between τ_B and τ_I

The following simple observation provides a basis of this work.

Proposition 3.4. The Bergmann tau-function τ_B from [10] and the isomonodromic tau-function τ_I are related as follows:

$$\tau_I = (\tau_B)^{-1/2}. \tag{3.28}$$
□

Proof. Let H_m be the quadratic Hamiltonians from (2.13). Due to Lemmas 3.1 and 3.3, we have

$$\begin{aligned}
H_m &= \frac{1}{2} \sum_{n \neq m} \frac{V_{mn}^2}{\lambda_m - \lambda_n} = \frac{1}{2} \sum_{n \neq m} \gamma_{mn}^2 (\lambda_m - \lambda_n) \\
&= \frac{1}{8} \sum_{n \neq m} b_{mn}^2 (P_m, P_n)(\lambda_m - \lambda_n) = -\frac{1}{2} \sum_{n \neq m} (\lambda_m - \lambda_n) \partial_{\lambda_n} \mathcal{B}_m \\
&= -\frac{1}{2} \left(\lambda_m \sum_{n \neq m} \partial_{\lambda_n} \mathcal{B}_m - \sum_{n \neq m} \lambda_n \partial_{\lambda_n} \mathcal{B}_m \right) \\
&= \frac{1}{2} \sum_{n=1}^M \lambda_n \partial_{\lambda_n} \mathcal{B}_m = -\frac{1}{2} \mathcal{B}_m,
\end{aligned} \tag{3.29}$$

which proves (3.28). ∎

3.4   The Bergmann tau-function for coverings with arbitrary branching over the point at infinity

In [10], the Bergmann tau-function τ_B was explicitly calculated in cases of Hurwitz spaces $H_{0,N}(1,\ldots,1)$ and $\widehat{H}_{1,N}(1,\ldots,1)$. In higher genera (i.e., for the spaces $\widehat{H}_{g,N}(1,\ldots,1)$ with $g \geq 2$) in [10], expressions for the modulus square $|\tau_B|^2$ were found. (It should be

noted that in [10] the general situation of Hurwitz spaces of coverings with higher multiplicities of the finite branch points was investigated. This general case corresponds to *nonsemisimple* Frobenius manifolds which are not considered here.)

A slight modification of the proofs from [10] leads to the explicit formulas of the Bergmann tau-function of the Hurwitz spaces $H_{0,N}(k_1, \ldots, k_l)$ and $\widehat{H}_{1,N}(k_1, \ldots, k_l)$ of coverings with the branching of type $(k_1, \ldots, k_l)$ over the point at infinity. (Coverings from $H_{g,N}(1, \ldots, 1)$ considered in [10] have no branching over the point at infinity.)

First, consider the case $g = 0$. Let $[p : \mathcal{L} \to \mathbb{P}^1] \in H_{0,N}(k_1, \ldots, k_l)$. Let also $z : \mathcal{L} \to \mathbb{P}^1$ be a biholomorphic map such that $z(\infty_1) = \infty$ and

$$z(P) = \left[\lambda(P)\right]^{1/k_1} + O(1), \tag{3.30}$$

as $P \to \infty_1$, where $\lambda(P) = p(P)$.

Introduce the local parameter ζ_s near the point ∞_s with $s \geq 2$:

$$\zeta_s(P) = \lambda^{-1/k_s}(P). \tag{3.31}$$

The map z near the point ∞_s ($s \geq 2$) is a holomorphic function of ζ_s. Near the simple ramification point P_m, the map z is a holomorphic function of the local parameter x_m from (3.4). The next statement is a modification of [10, Theorem 6]. Its proof is essentially the same.

Proposition 3.5. The Bergmann tau-function on the Hurwitz space $H_{0,N}(k_1, \ldots, k_l)$ is given by the following expression:

$$\tau_B = \left\{ \frac{\displaystyle\prod_{s=2}^{l} \left(\frac{dz}{d\zeta_s}\Big|_{\zeta_s=0} \right)^{k_s+1}}{\displaystyle\prod_{m=1}^{M} \frac{dz}{dx_m}\Big|_{x_m=0}} \right\}^{1/12}. \tag{3.32}$$

$\square$

Now let $g = 1$ and $[p : \mathcal{L} \to \mathbb{P}^1] \in \widehat{H}_{1,N}(k_1, \ldots, k_l)$, where $\mathcal{L}$ is an elliptic Riemann surface. Let ω be a holomorphic (not necessarily normalized) differential on $\mathcal{L}$. Introduce the notation

$$\begin{aligned} h_s &= \frac{\omega\big(\zeta_s(P)\big)}{d\zeta_s(P)}\bigg|_{P=\infty_s}, \quad s = 1, \ldots, l, \\ f_m &= \frac{\omega\big(x_m(P)\big)}{dx_m(P)}\bigg|_{P=P_m}, \quad m = 1, \ldots, M. \end{aligned} \tag{3.33}$$

Let σ be the modulus of the elliptic surface $\mathcal{L}$. Define the Dedekind eta-function by

$$
\eta(\sigma) = \left\{ \frac{\mathrm{d}}{\mathrm{d}z} \Theta \begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \end{bmatrix} (z, \sigma) \bigg|_{z=0} \right\}^{1/3} . \tag{3.34}
$$

The next statement is a modification of [10, Theorem 5].

Proposition 3.6. The Bergmann tau-function on the Hurwitz space $\widehat{H}_{1,N}(k_1, \ldots, k_l)$ is given by

$$
\tau_B = \eta^2 \left\{ \frac{\prod\limits_{s=1}^{l} h_s^{k_s+1}}{\prod\limits_{m=1}^{M} f_m} \right\}^{1/12} . \tag{3.35}
$$

$\square$

Due to Riemann-Hurwitz formula the right-hand side of (3.35) is independent of normalization of the holomorphic differential ω.

Remark 3.7. The way to obtain (3.32) and (3.35) was somehow indirect in [10]. Namely, these formulas were deduced from the study of the appropriately regularized Dirichlet integral $\mathbb{S} = (1/2\pi) \int_{\mathcal{L}} |\phi_\lambda|^2$, where $e^\phi |d\lambda|^2$ is the flat metric on $\mathcal{L}$ obtained by projecting down the standard metric $|dz|^2$ on the universal covering $\tilde{\mathcal{L}}$. The derivatives of $\mathbb{S}$ with respect to the branch points can be expressed through the values of the Schwarzian connection at the branch points; this reveals a close link between $\mathbb{S}$ and the modulus of the Bergmann tau-function. On the other hand, the integral $\mathbb{S}$ admits an explicit calculation via the asymptotics of the flat metric near the branch points and the infinities of the sheets of the covering. Moreover, it admits a "holomorphic factorization," that is, it can be explicitly represented as the modulus square of some holomorphic function, which allows one to compute the Bergmann tau-function itself.

To prove relations (3.32) and (3.35) directly (i.e., without any use of Dirichlet integrals) remains an open problem.

4   G-**function of Frobenius manifolds related to Hurwitz spaces in genera** 0 **and** 1

4.1   The general formulas for the G-function

The following two theorems are immediate consequences of Propositions 3.4, 3.5, and 3.6.

Theorem 4.1. The G-function of the Frobenius manifold with an underlying Hurwitz space $H_{0,N}(k_1, \ldots, k_l)$ and whose Frobenius structure is given by a primary differential ϕ can be expressed as follows:

$$
G = \frac{1}{24} \ln \left\{ \frac{\prod\limits_{m=1}^{M} \frac{dz}{dx_m}\Big|_{x_m=0}}{\prod\limits_{s=2}^{l} \left(\frac{dz}{d\zeta_s}\Big|_{\zeta_s=0} \right)^{k_s+1} \left(\prod\limits_{m=1}^{M} \operatorname{Res}_{P_m} \frac{\phi^2}{d\lambda} \right)^{1/2}} \right\}. \tag{4.1}
$$

$\square$

Theorem 4.2. The G-function of the Frobenius manifold with an underlying Hurwitz space $\widehat{H}_{1,N}(k_1, \ldots, k_l)$ and whose Frobenius structure is given by a primary differential ϕ can be expressed as follows:

$$
G = \frac{1}{24} \ln \left\{ \frac{\prod\limits_{m=1}^{M} f_m}{\prod\limits_{s=1}^{l} h_s^{k_s+1} \left(\prod\limits_{m=1}^{M} \operatorname{Res}_{P_m} \frac{\phi^2}{d\lambda} \right)^{1/2}} \right\} - \ln \eta(\sigma). \tag{4.2}
$$

$\square$

4.2   Examples

4.2.1   The Frobenius manifold $M_{0;N}$. This manifold is isomorphic to the orbit space $\mathbb{C}^{N-1}/A_{N-1}$ of the Coxeter group A_{N-1} (see [3]).

In this case, $l = 1$ and the first factor at the denominator of (4.1) is absent. As a map z, we can take one given by (2.2), so

$$
\left(\prod_{m=1}^{M} \operatorname{Res}_{P_m} \frac{\phi^2}{d\lambda} \right)^{1/48} = \left(\prod_{m=1}^{M} \operatorname*{Res}_{x_m=0} \frac{\left[z'(x_m) \right]^2 dx_m}{2x_m} \right)^{1/48}
$$

$$
= \operatorname{const} \left(\prod_{m=1}^{M} \frac{dz}{dx_m}\Big|_{x_m=0} \right)^{1/24} \tag{4.3}
$$

and, therefore, $G = \operatorname{const}$.

4.2.2   The Frobenius manifold $M_{0;k,N-k}$. According to [4], this manifold is isomorphic to the orbit space of the extended affine Weyl group $\tilde{W}^k(A_{N-1})$.

In this case, $l = 2$ and z is given by (2.3). Using the equality $\zeta_2 = \lambda^{-1/(N-k)}$, we get

$$
\begin{aligned}
\frac{dz}{d\zeta_2} &= \frac{1}{\lambda'_z} \frac{d\lambda}{d\zeta_2} \\
&= \frac{1}{b_N + O(z)} \left(\frac{z}{\zeta_2}\right)^{N-k+1} \\
&= \frac{1}{b_N + O(z)} \left[b_N + O(z)\right]^{(N-k+1)/(N-k)}
\end{aligned}
\tag{4.4}
$$

as $z \to 0$ and

$$
\begin{aligned}
\frac{dz}{d\zeta_2}\bigg|_{\zeta_2 = 0} &= b_N^{1/(N-k)}, \\
\left(\frac{dz}{d\zeta_2}\bigg|_{\zeta_2 = 0}\right)^{(k_2+1)/24} &= b_N^{(N-k+1)/24(N-k)}.
\end{aligned}
\tag{4.5}
$$

Since $\phi = dz/z$, we have

$$
\begin{aligned}
\left(\prod_{m=1}^{N} \operatorname{Res}_{P_m} \frac{\phi^2}{d\lambda}\right)^{1/48} &= \left(\prod_{m=1}^{N} \operatorname*{Res}_{x_m=0} \frac{[z'(x_m)]^2 dx_m}{2 x_m [z(x_m)]^2}\right)^{1/48} \\
&= \text{const} \frac{\left(\prod_{m=1}^{M} \frac{dz}{dx_m}\bigg|_{x_m=0}\right)^{1/24}}{\left(\prod_{m=1}^{M} \gamma_m\right)^{1/24}},
\end{aligned}
\tag{4.6}
$$

where $\gamma_m = z(P_m)$ are the critical points of the map $\lambda(z)$.

On the other hand, $M = 2g + l + N - 2 = N$ and

$$
\begin{aligned}
\lambda'(z) &= k z^{k-1} + \cdots + \frac{(k-N)b_N}{z^{N-k+1}} \\
&= \frac{k \prod_{n=1}^{N} (z - \gamma_n)}{z^{N-k+1}}.
\end{aligned}
\tag{4.7}
$$

Therefore,

$$
k \prod_{m=1}^{M} \gamma_m = \pm(N-k)b_N
\tag{4.8}
$$

and (up to a constant independent of $\{\lambda_k\}$)

$$G = -\frac{1}{24}\frac{\ln b_N}{N-k} = -\frac{1}{24}t_N, \tag{4.9}$$

in agreement with the main result of [14].

4.2.3 The Frobenius manifold $\widehat{M}_{1,N}$. This manifold is isomorphic to the orbit space $\mathbb{C} \times \mathbb{C}^{N-1} \times \{\Im z > 0\}/J(A_{N-1})$ of the Jacobi group $J(A_{N-1})$ (see [1, 2, 3]).

In this case, $l = 1$ and $M = N + 1$. Following [1, 2], we start the enumeration of the flat coordinates from 0. We have

$$\left(\prod_{m=1}^{M} \mathrm{Res}_{P_m} \frac{\phi^2}{d\lambda}\right)^{1/48} = \mathrm{const}\left(\prod_{m=1}^{M} \frac{\omega(x_m(P))}{dx_m(P)}\bigg|_{P=P_m}\right)^{1/24}$$
$$= \left(\prod_{m=1}^{M} f_m\right)^{1/24}. \tag{4.10}$$

On the other hand, since $\zeta_1 = \lambda^{-1/N}$,

$$t_N = \mathrm{Res}_{z=0}\left(z[\lambda(z)]^{-(N-1)/N}d\lambda(z)\right)$$
$$= \mathrm{Res}_{\zeta_1=0}\left(z(\zeta_1)\frac{d\zeta_1}{\zeta_1^2}\right) = z'(\zeta_1)|_{\zeta_1=0} = h_1, \tag{4.11}$$
$$G = -\ln\eta(t_0) - \frac{N+1}{24}t_N,$$

which proves the conjecture from [14].

5 Some remarks on higher genus case

Here we give a formula for the modulus square of the tau-function of Frobenius manifolds related to the Hurwitz spaces $H_{g,N}(k_1, \ldots, k_l)$ with $g \geq 2$. From this formula, one can derive an expression for the real part of the corresponding G-function. For simplicity, we consider only the case $k_1 = \cdots = k_l = 1$; the results in the general case differ insignificantly.

If the covering $\mathcal{L}$ has genus $g > 1$, then it is biholomorphically equivalent to the quotient space $\mathbb{H}/\Gamma$, where $\mathbb{H} = \{z \in \mathbb{C} : \Im z > 0\}$; Γ is a strictly hyperbolic Fuchsian group. Denote by $\pi_F : \mathbb{H} \to \mathcal{L}$ the natural projection. Let x be a local parameter on $\mathcal{L}$. Introduce

the standard metric of the constant curvature -1 on $\mathcal{L}$:

$$e^{\chi(x,\bar{x})}|dx|^2 = \frac{|dz|^2}{|\Im z|^2}, \tag{5.1}$$

where $z \in \mathbb{H}$, $\pi_F(z) = P$, and $x = x(P)$.

Denote by $\zeta = 1/\lambda$ the local coordinate in a neighborhood of the infinity of any sheet of covering $\mathcal{L}$. Introduce functions $\chi^{\text{ext}}(\lambda, \bar{\lambda})$, $\chi^{\text{int}}(x_m, \bar{x}_m)$, $m=1,\ldots,M$, and $\chi_k^{\infty}(\zeta, \bar{\zeta})$, $k = 1,\ldots,N$ by specifying $x = \lambda$, $x = x_m$, and $x = \zeta$ (in a neighborhood of the point at infinity of the kth sheet) in (5.1), respectively.

Consider the following domain on the kth sheet of $\mathcal{L}$: $\mathcal{L}_\rho^k = \{\lambda \in \mathcal{L}^k : \forall m \, |\lambda - \lambda_m| > \rho, |\lambda| < 1/\rho\}$, where λ_m are all the branch points which belong to the kth sheet $\mathcal{L}^k$ of the covering $\mathcal{L}$. (The sheet $\mathcal{L}^k$ can be considered as a copy of the Riemann sphere $\mathbb{P}^1$ with appropriate cuts between the branch points, and the domain $\mathcal{L}_\rho^k$ is obtained from $\mathcal{L}^k$ by deleting small discs around branch points belonging to this sheet and the disc around infinity.)

The function $\chi_k^{\text{ext}} : \mathcal{L}^k \to \mathbb{R}$ is smooth in the domain $\mathcal{L}_\rho^k$ for any sufficiently small $\rho > 0$. This function has finite limits at the cuts (except the endpoints which are the branch points); at the branch points and at the infinity there are the following asymptotics:

$$\left|\partial_\lambda \chi_k^{\text{ext}}(\lambda, \bar{\lambda})\right|^2 = \frac{1}{4}|\lambda - \lambda_m|^{-2} + O\left(|\lambda - \lambda_m|^{-3/2}\right) \tag{5.2}$$

as $\lambda \to \lambda_m$, and

$$\left|\partial_\lambda \chi_k^{\text{ext}}(\lambda, \bar{\lambda})\right|^2 = 4|\lambda|^{-2} + O\left(|\lambda|^{-3}\right) \tag{5.3}$$

as $\lambda \to \infty$. We define the "truncated" integral T_ρ by

$$T_\rho = \sum_{k=1}^{N} \int_{\mathcal{L}_\rho^k} \frac{\left|\partial_\lambda \chi_k^{\text{ext}}\right|^2 |d\lambda \wedge d\bar{\lambda}|}{2}. \tag{5.4}$$

Then there exists the finite limit

$$\text{reg} \int_{\mathcal{L}} \frac{\left(|\chi_\lambda|^2 + e^\chi\right)|d\lambda \wedge d\bar{\lambda}|}{2}$$

$$= \lim_{\rho \to 0} \left(T_\rho + \sum_{k=1}^{N} \int_{\mathcal{L}^k} \frac{e^{\chi_k^{\text{ext}}}|d\lambda \wedge d\bar{\lambda}|}{2} + (8N + 4M)\pi \ln \rho\right). \tag{5.5}$$

Define the function $\mathbb{S}_F$ by

$$
\begin{aligned}
\mathbb{S}_F(\lambda_1, \ldots, \lambda_M) = \frac{1}{24\pi} & \left\{ \operatorname{reg} \int_{\mathcal{L}} \frac{\left(|\chi_\lambda|^2 + e^\chi \right) |d\lambda \wedge d\bar{\lambda}|}{2} \right\} \\
& + \frac{1}{3} \sum_{n=1}^{M} \chi^{\mathrm{int}}(x_n) \Big|_{x_n = 0} - \frac{1}{3} \sum_{k=1}^{N} \chi_k^\infty(\zeta) \Big|_{\zeta = 0}
\end{aligned}
\tag{5.6}
$$

and introduce the determinant of the Laplacian operator (in the Poincaré metric) $\det \Delta = \exp\{-\zeta'(0)\}$, where $\zeta(s)$ is the zeta-function of the Laplacian on the Riemann surface $\mathcal{L}$.

Let $\mathbb{B}$ be the matrix of $\mathfrak{b}$-periods of the Riemann surface $\mathcal{L}$. The following theorem is a consequence of [10, Theorem 9] and Lemma 3.1.

Theorem 5.1. Let $\mathfrak{g} \geq 2$. The modulus square of the isomonodromic tau-function on $\widehat{H}_{\mathfrak{g},N}(1, \ldots, 1)$ has the following representation:

$$
|\tau_I|^2 = e^{\mathbb{S}_F} \frac{(\det \mathfrak{J}\mathbb{B})^{1/2}}{(\det \Delta)^{1/2}}.
\tag{5.7}
$$
$\square$

Remark 5.2. At the moment, we do not know the explicit holomorphic factorization (similar to that in genera 0 and 1) of the right-hand side of (5.7). Finding such a factorization seems to be of great interest.

Let M_ϕ be the Frobenius manifold with underlying Hurwitz space $\widehat{H}_{\mathfrak{g},N}(1, \ldots, 1)$ and the Frobenius structure given by a primary differential ϕ. From Theorem 5.1, it follows that the real part of the G-function of M_ϕ is given by

$$
\operatorname{Re} G = \frac{1}{2} \mathbb{S}_F + \frac{1}{4} \ln \frac{(\det \mathfrak{J}\mathbb{B})}{(\det \Delta)} - \frac{1}{48} \ln \left| \prod_{m=1}^{M} \operatorname{Res}_{P_m} \frac{\phi^2}{d\lambda} \right|.
\tag{5.8}
$$

Acknowledgments

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