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$1/N^2$ Correction to Free Energy in Hermitian Two-Matrix Model

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Abstract. Using the loop equations we find an explicit expression for genus 1 correction in hermitian two-matrix model in terms of holomorphic objects associated to spectral curve arising in large N limit. Our result generalises known expression for F^1 in hermitian one-matrix model. We discuss the relationship between F^1, Bergman tau-function on Hurwitz spaces, G-function of Frobenius manifolds and determinant of Laplacian over spectral curve.

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In this Letter, we derive an explicit formula for the $1/N^2$ correction to free energy F of hermitian two-matrix model:

$$\mathrm{e}^{-N^2 F} := \int \mathrm{d}M_1 \mathrm{d}M_2 \mathrm{e}^{-N\mathrm{tr}\{V_1(M_1)+V_2(M_2)-M_1M_2\}}. \tag{1}$$

It is hard to overestimate the interest to random matrix models in modern physics and mathematics; we just mention their appearance in statistical physics, condensed matter and 2d quantum gravity (see e.g. [1]) . The expansion $F = \sum_{G=0}^{\infty} N^{-2G} F^G$ (N is the matrix size) in hermitian matrix models is one of the cornerstones of the theory, due to its clear physical interpretation as topological expansion of the functional integral, which appears in $N \to \infty$ limit; in statistical physics the term F^G plays the role of free energy for statistical physics model on genus G Riemann surface. From the whole zoo if the random matrices one of the simplest is the hermitian one-matrix model with partition function $\mathrm{e}^{-N^2 F} = \int \mathrm{d}M \mathrm{e}^{-N\mathrm{tr}V(M)}$ (V is a polynomial), which can be used as testing ground for the methods applied in more general situations of two- and multi-matrix models. The most rigorous way to compute the $1/N^2$ expansion for both one-matrix and two-matrix models is based on the loop equations. The loop equations follow from the reparametrization invariance of matrix integrals; for one-matrix case the loop equations were used to compute F^1 (see [2]). Later the loop equations were written down for the case of two-matrix model [3,4] and F^1 was found for the case when the spectral curve has genus zero and one [4]; for arbitrary genus of spectral curve of two-matrix model only the leading term F^0 is known (see [5]).

Let us write down polynomials V_1 and V_2 in the form $V_1(x)=\sum_{k=1}^{d_1+1}\frac{u_k}{k}x^k$ and $V_2(y)=\sum_{k=1}^{d_2+1}\frac{v_k}{k}y^k$. It is sometimes convenient to think of V_1 and V_2 as infinite formal power expansions: $V_1(x)=\sum_{k=1}^{\infty}\frac{u_k}{k}x^k$, $V_2(y)=\sum_{k=1}^{\infty}\frac{v_k}{k}y^k$, where coefficients u_k vanish for $k\geqslant d_1+2$, and v_k vanish for $k\geqslant d_2+2$. According to this point of view the operators of differentiation with respect to coefficients of V_1 and V_2 have the following meaning (see [5]):

$$\begin{aligned}\left.\frac{\delta}{\delta V_1(x)}\right|_x &:= \left.\left\{\sum_{k=1}^{\infty}x^{-k-1}k\partial_{u_k}\right\}\right|_{u_k=0,k\geqslant d_1+2},\\ \left.\frac{\delta}{\delta V_2(y)}\right|_y &:= \left.\left\{\sum_{k=1}^{\infty}y^{-k-1}k\partial_{v_k}\right\}\right|_{v_k=0,k\geqslant d_1+2}.\end{aligned}\tag{2}$$

As it was discussed in detail in [5], (3) is a formal notation which makes sense only order by order in the infinite power series expansion; it allows to write an infinite number of equations at once. Consider the resolvents $\mathcal{W}(x)=\frac{1}{N}\left\langle\mathrm{tr}\frac{1}{x-M_1}\right\rangle$ and $\tilde{\mathcal{W}}(y)=\frac{1}{N}\left\langle\mathrm{tr}\frac{1}{y-M_2}\right\rangle$. The free energy of two-matrix model (1) satisfies the following equations with respect to coefficients of polynomials V_1 and V_2:

$$\frac{\delta F}{\delta V_1(x)}=\mathcal{W}(x),\qquad \frac{\delta F}{\delta V_2(y)}=\tilde{\mathcal{W}}(y).\tag{3}$$

The equations (3) were solved in [5] in the zeroth order assuming the finite-gap structure of distribution of eigenvalues of M_1 (and, *a posteriori*, also of M_2) as $N\to\infty$. Here we find the next coefficient F^1, using the loop equations. The spectral curve $\mathcal{L}$ is defined by the following equation, which arises in the zeroth order approximation:

$$\mathcal{E}^0(x,y):=(V_1{}'(x)-y)(V_2{}'(y)-x)-\mathcal{P}^0(x,y)+1=0\tag{4}$$

where the polynomial of two variables $\mathcal{P}^0(x,y)$ is the zeroth order term in $1/N^2$ expansion of the polynomial

$$\mathcal{P}(x,y):=\frac{1}{N}\left\langle\mathrm{tr}\frac{V_1{}'(x)-V_1{}'(M_1)}{x-M_1}\frac{V_2{}'(y)-V_2{}'(M_2)}{y-M_2}\right\rangle;\tag{5}$$

the point $P\in\mathcal{L}$ of the curve is the pair of complex numbers (x,y) satisfying (4) (on the "physical" sheet the equation of spectral curve (4) defines an implicit function $y(x)$, which gives the zeroth order approximation to $V_2{}'(x)-\mathcal{W}(x)$). The spectral curve (4) comes together with two meromorphic functions $f(P)=x$ and $g(P)=y$, which project it down to x and y-planes, respectively. These functions have poles only at two points of $\mathcal{L}$, called ∞_f and ∞_g: at ∞_f function $f(P)$

has simple pole, and function $g(P)$ – pole of order d_1 with singular part equal to $V_1{}'(f(P))$. At ∞_g the function $g(P)$ has simple pole, and function $f(P)$ – pole of order d_2 with singular part equal to $V_2{}'(g(P))$. In addition, in our normalization of partition function (1) we have the asymptotics $\mathcal{W}(x)\sim_{x\to\infty} 1/x+\cdots$ and $\tilde{\mathcal{W}}(y)\sim_{y\to\infty} 1/y+\cdots$, which imply [5] $\mathrm{Res}_{\infty_f} g\,\mathrm{d}f=1$ and $\mathrm{Res}_{\infty_g} f\mathrm{d}g=1$, respectively. Therefore, one gets the moduli space $\mathcal{M}$ of triples $(\mathcal{L}, f, g)$, where functions f and g have this pole structure. The natural coordinates on this moduli space are coefficients of polynomials V_1 and V_2 and g numbers, called "filling fractions" $\epsilon_\alpha = \frac{1}{2\pi i}\oint_{a_\alpha} g\,\mathrm{d}f$, where (a_α, b_α) is some basis of canonical cycles on $\mathcal{L}$. The additional constraints which should be imposed *a posteriori* to make the "filling fractions" dependent on coefficients of polynomials V_1 and V_2 are (according to one-matrix model experience, these conditions correspond to non-tunneling between different intervals of eigenvalues support): $\oint_{b_a} g\,\mathrm{d}f=0$. Denote the zeros of differential $\mathrm{d}f$ by $P_1,\dots,P_{m_1}$ $(m_1=d_2+2g+1)$ (these points play the role of ramification points if we realize $\mathcal{L}$ as branched covering by function $f(P)$); their projections on x-plane are the branch points, which we denote we denote by $\lambda_j := f(P_j)$. The zeros of the differential $\mathrm{d}g$ (the ramification points if we consider $\mathcal{L}$ as covering defined by function $g(P)$) we denote by $Q_1,\dots,Q_{m_2}$ $(m_2=d_1+2g+1)$; there projections on y-plane (the branch points) we denote by $\mu_j := g(Q_j)$. We shall assume that our potentials V_1 and V_2 are generic i.e. all zeros of differentials $\mathrm{d}f$ and $\mathrm{d}g$ are simple, and none of the zeros of $\mathrm{d}f$ coincides with a zero of $\mathrm{d}g$. It is well-known (see for instance [5]) how to express all standard algebro-geometrical objects on $\mathcal{L}$ in terms of the previous data. In particular, the canonical meromorphic bidifferential $B(P,Q)=d_P d_Q \ln E(P,Q)$ ($E(P,Q)$ is the prime-form) can be represented as follows:

$$B(P,Q)=\frac{\delta g(P)}{\delta V_1(f(Q))}\Big|_{f(Q)}\mathrm{d}f(P)\mathrm{d}f(Q) \tag{6}$$

This bidifferential has the following behaviour near diagonal $P\to Q$: $B(P,Q)=\{(z(P)-z(Q))^{-2}+\frac{1}{6}S_B(P)+o(1)\}\mathrm{d}z(P)\mathrm{d}z(Q)$, where $z(P)$ is some local coordinate; $S_B(P)$ is the Bergman projective connection . Consider also the four-differential $D(P,Q)=d_P d_Q^3 \ln E(P,Q)$, which has on the diagonal the pole of 4th degree: $D(P,Q)=\{6(z(P)-z(Q))^{-4}+O(1)\}\mathrm{d}z(P)(\mathrm{d}z(Q))^3$. From $B(P,Q)$ and $D(P,Q)$ it is easy to construct meromorphic normalized (all a-periods vanish) 1-forms on $\mathcal{L}$ with single pole; in particular, if the pole coincides with ramification point P_k, the natural local parameter near P_k is $x_k(P)=\sqrt{f(P)-\lambda_k}$; then $B(P,P_k):=\frac{B(P,Q)}{\mathrm{d}x_k(Q)}\Big|_{Q=P_k}$ and $D(P,P_k):=\frac{D(P,Q)}{(\mathrm{d}x_k(Q))^3}\Big|_{Q=P_k}$ are meromorphic normalized 1-forms on $\mathcal{L}$ with single pole at P_k and the following singular parts:

$$\begin{aligned} B(P,P_k)&=\left\{\frac{1}{[x_k(P)]^2}+\frac{1}{6}S_B(P_k)+o(1)\right\}\mathrm{d}x_k(P);\\ D(P,P_k)&=\left\{\frac{6}{[x_k(P)]^4}+O(1)\right\}\mathrm{d}x_k(P)\end{aligned} \tag{7}$$

as $P \to P_k$, where $S_B(P_k)$ is the Bergman projective connection computed at the branch point P_k with respect to the local parameter $x_k(P)$.

Equations (3) in order $1/N^2$ look as follows (we write only equations with respect to V_1):

$$\frac{\delta F^1}{\delta V_1(f(P))} = -Y^1(P) \tag{8}$$

where the Y^1 is the (taken with minus sign) $1/N^2$ contribution to the resolvent $\mathcal{W}$. The function Y^1 can be computed using the loop equations [4] and the "normalization conditions"

$$\oint_{a_\alpha} Y^1(P)\mathrm{d}f(P) = 0 \tag{9}$$

over all basic a-cycles (these conditions mean that the "filling fractions" do not have the $1/N^2$ correction). We introduce also the polynomial

$$\mathcal{E}(x, y) := (V_1(x) - y)(V_2(y) - x) - \mathcal{P}(x, y) + 1, \tag{10}$$

the function $\mathcal{U}(x, y)$, which is a polynomial in y and rational function in x:

$$\mathcal{U}(x, y) := \frac{1}{N}\left\langle \mathrm{tr}\frac{1}{x - M_1}\frac{V_2{}'(y) - V_2{}'(M_2)}{y - M_2}\right\rangle, \tag{11}$$

and rational function $\mathcal{U}(x, y, z)$:

$$\mathcal{U}(x, y, z) := \frac{\delta \mathcal{U}(x, y)}{\delta V_1(z)} = \left\langle \mathrm{tr}\frac{1}{x - M_1}\frac{V_2{}'(y) - V_2{}'(M_2)}{y - M_2}\mathrm{tr}\frac{1}{z - M_1}\right\rangle - N^2\mathcal{U}(x, y)\mathcal{W}(z). \tag{12}$$

Then the loop equation looks as follows:

$$\mathcal{U}(x, y) = x - V_2{}'(y) + \frac{\mathcal{E}(x, y)}{y - Y(x)} - \frac{1}{N^2}\frac{\mathcal{U}(x, y, x)}{y - Y(x)}, \tag{13}$$

where $Y(x) := V_1{}'(x) - \mathcal{W}(x)$; it arises as a corollary of reparametrization invariance of the matrix integral (1) [4]. To use the loop equation effectively we need to consider the $1/N^2$ expansion of all of their ingredients. In this way we get the following expression for Y^1:

$$Y^1(P)\mathrm{d}f(P) = \frac{\mathcal{P}^1(f(P), g(P))\mathrm{d}f(P)}{\mathcal{E}_y^0(f(P), g(P))} + \sum_{Q \neq P\,:\, f(Q) = f(P)} \frac{B(P, Q)}{df(Q)}\frac{1}{g(P) - g(Q)}, \tag{14}$$

where $\mathcal{E}_y^0(x, y)$ means partial derivative with respect to the second argument. All ingredients of (14) arise already in the leading term, except $\mathcal{P}^1$. From (5) we see that $\mathcal{P}(x, y)$ is a polynomial of degree $d_1 - 1$ with respect to x and $d_2 - 1$ with respect to y; moreover, the coefficient in front of $x^{d_1 - 1}y^{d_2 - 1}$ does not have $1/N^2$

correction. Now, assuming that $\mathcal{L}$ is a non-singular algebraic curve (i.e. it has the "maximal" genus equal d_1d_2-1), we can claim that the differential $\mathrm{d}f$ vanishes only at the branch points, where $\mathcal{E}_y^0(f(P), g(P))=0$. Thus we can conclude that if the spectral curve $\mathcal{L}$ is non-singular, the one-form $Y^1(P)\mathrm{d}f(P)$ is non-singular on $\mathcal{L}$ outside of the branch points P_m, where it has poles of order 4. Moreover, the first term in (14) is non-singular on $\mathcal{L}$ (the first order zeros of $\mathcal{E}_y^0$ at the branch points are cancelled by first order zeros of $\mathrm{d}f(P)$ at these points). If the spectral curve is singular i.e. the genus of $\mathcal{L}$ is smaller than the maximal genus, the non-singularity of $Y^1(P)\mathrm{d}f(P)$ outside of the branch points is suggested by physical consideration: since at the double points one does not have any eigenvalues of matrices M_1 and M_2, there is no physical reason to expect that in large N limit the resolvents are singular at these points (similar assumption is made in one-matrix model case, too [2]).

The form of singular parts at P_m allows to determine $Y^1(P)\mathrm{d}f(P)$ completely in terms of differentials $B(P, P_k)$ and $D(P, P_k)$ if we take into account the absence of $1/N^2$ correction to the "filling fractions" (9); the result looks as follows:

$$Y^{(1)}(P)\mathrm{d}f(P)=\sum_{k=1}^{m_1}\left\{-\frac{1}{96g'(P_k)}D(P, P_k)+\left[\frac{g'''(P_k)}{96g'^2(P_k)}-\frac{S_B(P_k)}{24g'(P_k)}\right]B(P, P_k)\right\} \tag{15}$$

Then the solution of (8), (15), which is symmetric with respect to the projection change (and, therefore, satisfies also equations (3) with respect to V_2), looks as follows:

$$F^1=\frac{1}{24}\ln\left\{\tau_f^{12}(v_{d_2+1})^{1-\frac{1}{d_2}}\prod_{k=1}^{m_1}\mathrm{d}g(P_k)\right\}+\frac{d_2+3}{24}\ln d_2 \tag{16}$$

where τ_f is the so-called Bergman tau-function on Hurwitz space [7], which satisfies the following system of equations with respect to the branch points λ_k:

$$\frac{\partial}{\partial\lambda_k}\ln\tau_f=-\frac{1}{12}S_B(P_k)\,; \tag{17}$$

In derivation of (15) we have used the following variational formulas, which can be easily proved in analogy to Rauch variational formulas:

$$-\frac{\delta\lambda_k}{\delta V_1(f(P))}g'(P_k)\mathrm{d}f(P)=B(P, P_k), \tag{18}$$

$$\left.\frac{\delta\{g'(P_k)\}}{\delta V_1(f(P))}\right|_{f(P)}\mathrm{d}f(P)=\frac{1}{4}\left\{D(P, P_k)-\frac{g'''(P_k)}{g'(P_k)}B(P, P_k)\right\} \tag{19}$$

The Bergman tau-function (17) appears in many important problems: it coincides with isomonodromic tau-function of Hurwitz Frobenius manifolds [6], and gives the main contribution to G-function (solution of Getzler equation) of these Frobenius manifolds; it gives the most non-trivial term in isomonodromic tau-function of Riemann–Hilbert problem with quasi-permutation monodromies. Finally, its modulus square essentially coincides with determinants of Laplace operator in metrics with conical singularities over Riemann surfaces [7]. The solution of the system (17) looks as follows [7]. Introduce the divisor $(\mathrm{d}f) = -2\infty_f - (d_2+1)\infty_g + \sum_{k=1}^{m_1} P_k := \sum_{k=1}^{m_1+2} r_k D_k$. Choose some initial point $P \in \hat{\mathcal{L}}$ and introduce corresponding vector of Riemann constants K^P and Abel map $\mathcal{A}_\alpha(Q) = \int_P^Q w_\alpha$ (w_α form the basis of normalized holomorphic 1-forms on $\mathcal{L}$). Since some points of divisor $(\mathrm{d}f)$ have multiplicity 1, we can always choose the fundamental cell $\hat{\mathcal{L}}$ of the universal covering of $\mathcal{L}$ in such a way that $\mathcal{A}((\mathrm{d}f)) = -2K^P$ (for an arbitrary choice of fundamental domain these two vectors coincide only up to an integer combination of periods of holomorphic differentials), where the Abel map is computed along the path which does not intersect the boundary of $\hat{\mathcal{L}}$. The main ingredient of the Bergman tau-function is the following holomorphic multivalued $(1-g)g/2$-differential $\mathcal{C}(P)$ (the higher genus analog of Dedekind eta-function) on $\mathcal{L}$:

$$\mathcal{C}(P) := \frac{1}{W(P)} \sum_{\alpha_1,\dots,\alpha_g=1}^{g} \frac{\partial^g \Theta(K^P)}{\partial z_{\alpha_1} \dots \partial z_{\alpha_g}} w_{\alpha_1}(P) \dots w_{\alpha_g}(P). \tag{20}$$

where $W(P) := \det_{1 \leqslant \alpha,\beta \leqslant g} ||w_\beta^{(\alpha-1)}(P)||$ denotes the Wronskian determinant of holomorphic differentials at point P; K^P is the vector of Riemann constants with basepoint P; Θ is the g-dimensional theta-function built from matrix of b-periods of the curve $\mathcal{L}$. Introduce the quantity $\mathcal{Q}$ defined by

$$\mathcal{Q} = [\mathrm{d}f(P)]^{\frac{g-1}{2}} \mathcal{C}(P) \prod_{k=1}^{m_2+2} [E(P, D_k)]^{\frac{(1-g)r_k}{2}}; \tag{21}$$

which is independent of the point $P \in \mathcal{L}$. Then the Bergman tau-function (17) of Hurwitz space is given by the following expression [7]:

$$\tau_f = \mathcal{Q}^{2/3} \prod_{k,l=1\ k<l}^{m_2+2} [E(D_k, D_l)]^{\frac{r_k r_l}{6}}; \tag{22}$$

together with (16) this gives the answer for $1/N^2$ correction in two-matrix model. If τ_f and τ_g are Bergman tau-functions (17) corresponding to divisors $(\mathrm{d}f)$ and $(\mathrm{d}g)$, respectively, then

$$\left(\frac{\tau_f}{\tau_g}\right)^{12} = C \frac{(u_{d_1+1})^{1-\frac{1}{d_1}}}{(v_{d_2+1})^{1-\frac{1}{d_2}}} \frac{\prod_k \mathrm{d}f(Q_k)}{\prod_k \mathrm{d}g(P_k)} \tag{23}$$

where $C = d_1^{d_1+3}/d_2^{d_2+3}$ is a constant independent of moduli parameters. Using the transformation (23) of the Bergman tau-function under projection change, we find that the solution expression (16) for F^1 satisfies also the necessary equations with respect to V_2. This could be considered as a confirmation of consistency of our computation. Derivatives of function F^1 (16) with respect to the filling fractions look as follows:

$$\frac{\partial F^1}{\partial \epsilon_\alpha} = -\oint_{b_\alpha} Y^1(P)\mathrm{d}f(P); \tag{24}$$

these equations are $1/N^2$ counterparts of Seiberg-Witten type equations for F^0 (see for example [4,5,8]).

Genus zero ("one-cut") case. For genus zero the expression for the Bergman tau-function (22) can be rewritten in terms of the uniformization map $z(P)$ of the Riemann surface $\mathcal{L}$ to the Riemann sphere, satisfying the condition $z(P) = l + O(1)$ as $P \to \infty_f$. The formula for τ_f looks as follows (see (3.32), (4.5) in [15]):

$$\tau_f^{12} = (v_{d_2+1})^{1+1/d_2} \prod_{k=1}^{d_2+1} \frac{\mathrm{d}x_k}{\mathrm{d}z}(P_k).$$

This expression can be derived from (22) using the formula for the prime-form on $\mathcal{L}$ obtained as pull-back of the prime-form on the Riemann sphere: $E(P,Q) = \frac{z(P)-z(Q)}{\sqrt{\mathrm{d}z(P)}\sqrt{\mathrm{d}z(Q)}}$. Substituting this formula into (16) and using the chain rule $\frac{\mathrm{d}g}{\mathrm{d}x_k}(P_k)\frac{\mathrm{d}x_k}{\mathrm{d}z}(P_k) = \frac{\mathrm{d}g}{\mathrm{d}z}(P_k)$, we rewrite (16) as follows:

$$F^1 = \frac{1}{24}\ln\left\{ v_{\mathrm{d}_2+1}^2 \prod_{k=1}^{\mathrm{d}_2+1} \frac{\mathrm{d}g}{\mathrm{d}z}(P_k)\right\} + C$$

where C is a constant, in agreement with the formula previously obtained in [4].

Elliptic spectral curve ("two-cut" case). Denote the period of the spectral curve $\mathcal{L}$ by σ. The Bergman tau-function (22) for elliptic covering with multiplicities of points at infinity equal to 1 and d_2 can be represented as follows ([15], (3.35)):

$$\tau_f^{12} = \eta^{24}(\sigma)\left(\frac{w}{d(f^{-1})}(\infty_f)\right)^2 \left(\frac{w}{d(f^{-1/d_2})}(\infty_g)\right)^{d_2+1} \prod_{k=1}^{d_2+3} \frac{\mathrm{d}x_k}{w}(P_k) \tag{25}$$

where $\eta(\sigma) = [\vartheta_1'(0,\sigma)]^{1/3}$ is the Dedekind eta-function; w is an arbitrary holomorphic one-form on $\mathcal{L}$ (it is easy to see that (25) remains invariant if w is multiplied by an arbitrary constant). For simplicity we can normalize w such that at ∞_g we get $w(P) = d(f^{-1/d_2}(P))[1+o(1)]$. Under this normalization we get the following formula for F^1:

$$F^1 = \ln\eta(\sigma) + \frac{1}{24}\ln\left\{ (v_{d_2+1})^{1+1/d_2}\left(\frac{w}{d(f^{-1})}(\infty_f)\right)^2 \prod_{k=1}^{d_2+3} \frac{\mathrm{d}g}{w}(P_k)\right\} + C, \tag{26}$$

which is new; this function looks different (although is, in fact, the same) from the expression previously obtained in [4]. The expression obtained in [4] can be derived by straightforward specialization of the formula (16) to genus 1 case using the following expression for the prime-form in genus one: $E(P,Q)=\frac{\vartheta_1'(z(P)-z(Q))}{\vartheta_1'(0)\sqrt{\mathrm{d}z(P)}\sqrt{\mathrm{d}z(Q)}}$, where $z(P)$ is the uniformization map of the curve $\mathcal{L}$ to the torus with periods 1 and σ (in the elliptic case the differential $C(P)$ does not depend on P and equals $\vartheta_1'(0)$).

One-matrix model. If potential V_2 is quadratic, integration with respect to M_2 in (1) can be performed explicitly, and the free energy (16) gives rise to the free energy of one-matrix model. The spectral curve $\mathcal{L}$ in this case becomes hyperelliptic, and the formula (16) gives, using the expression for τ_f obtained in [9]:

$$F^1 = \frac{1}{24}\ln\left\{\Delta^3\,(\det \mathbf{A})^{12}\prod_{k=1}^{2g+2} g'(\lambda_k)\right\} \tag{27}$$

where λ_k, $k=1,\dots,2g+2$ are branch points of $\mathcal{L}$; Δ is their Wronskian determinant; $\mathbf{A}$ is the matrix of a-periods of non-normalized holomorphic differentials on $\mathcal{L}$; this agrees with previous results [2].

F^1, isomonodromic tau-function and G-function of Frobenius manifolds. The genus 1 correction to free energy in topological field theories is given by so-called G-function, which for an arbitrary m-dimensional Frobenius manifold related to Hurwitz space looks as follows [6,7,10]: $\exp\{G\}=\tau_f^{-1/2}\prod_{k=1}^{m}\{\mathrm{res}|_{P_k}\frac{\varphi^2}{\mathrm{d}f}\}^{-1/48}$, where τ_f is the Bergman tau-function, φ is an "admissible" one-form on underlying Riemann surface. If, trying to build an analogy with our formula (16) for F^1, we formally choose $\phi(P)=\mathrm{d}g(P)$, the formula for G-function coincides with (16) up to small details like sign, additive constant, and the highest coefficient of polynomial V_2. However, the differential $\mathrm{d}g$ is not admissible, and, therefore, does not really correspond to any Frobenius manifold; therefore, the true origin of his analogy is still unclear at the moment.

F^1 and determinant of Laplace operator. Existence of close relationship between F^1 and determinant of Laplace operator was suggested by several authors (see e.g. [11] for hermitial one-matrix model, [4] for hermitian two-matrix model and, finally, [14] for normal two-matrix model with simply-connected support of eigenvalues). After an appropriate regularization the (formal) determinant of Laplace operator Δ^f over $\mathcal{L}$ in the singular metric $|\mathrm{d}f(P)|^2$ is given by the expression [13] $\det\Delta^f = C\,\mathcal{A}\{\det\Im\mathbf{B}\}|\tau_f|^2$, where $\mathcal{A}$ is a regularized area of $\mathcal{L}$, $\mathbf{B}$ is the matrix of b-periods of $\mathcal{L}$, C is a constant. In the "physical" case of real coefficients of V_1 and V_2 and real filling fractions the empirical expression for $\ln\{\det\Delta^f\}$ differs from our expression (16) by several explicit terms. Therefore, the relationship between hermitian and normal two-matrix models on the level of F^1 seems to be not as straightforward as on the level of functions F^0 [5,14].

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