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# Determinant of the Laplacian on Tori of Constant Positive Curvature with one Conical Point

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*Abstract.* We find an explicit expression for the zeta-regularized determinant of (the Friedrichs extensions of) the Laplacians on a compact Riemann surface of genus one with conformal metric of curvature 1 having a single conical singularity of angle $4\pi$.

## 1 Introduction

Let $X$ be a compact Riemann surface of genus one and let $P \in X$. According to [1, Cor. 3.5.1], there exists at most one conformal metric on $X$ of constant curvature 1 with a (single) conical point of angle $4\pi$ at $P$. The following simple construction shows that such a metric, $m(X, P)$, in fact always exists (and, due to [1], is unique).

Consider the spherical triangle $T = \{(x_1, x_2, x_3) \in S^2 \subset \mathbb{R}^3 : x_1 \geq 0, x_2 \geq 0, x_3 \geq 0\}$ with all three angles equal to $\pi/2$. Gluing two copies of $T$ along their boundaries, we get the Riemann sphere $\mathbb{C}P^1$ with metric $m$ of curvature 1 and three conical points $P_1, P_2, P_3$ of conical angle $\pi$. Consider the two-fold covering

$$\mu\colon X(Q) \longrightarrow \mathbb{C}P^1$$

ramified over $P_1$, $P_2$, $P_3$ and some point $Q \in \mathbb{C}P^1 \setminus \{P_1, P_2, P_3\}$. Lifting the metric $m$ from $\mathbb{C}P^1$ to the compact Riemann surface $X(Q)$ of genus one via $\mu$, one gets the metric $\mu^* m$ on $X(Q)$ that has curvature 1 and the unique conical point of angle $4\pi$ at the preimage $\mu^{-1}(Q)$ of $Q$. Clearly, any compact surface of genus one is (biholomorphically equivalent to) $X(Q)$ for some $Q \in \mathbb{C}P^1 \setminus \{P_1, P_2, P_3\}$. Now let $X$ be an arbitrary compact Riemann surface of genus one and let $P$ be any point of $X$. Take $Q \in \mathbb{C}P^1$ such that $X = X(Q)$ and consider the automorphism $\alpha\colon X \to X$ (the translation) of $X$ sending $P$ to $\mu^{-1}(Q)$. Then

$$m(X, P) = \alpha^*\big(\mu^*(m)\big) = (\mu \circ \alpha)^*(m).$$

Introduce the scalar (Friedrichs) self-adjoint Laplacian $\Delta(X, P) := \Delta^{m(X,P)}$ on $X$ corresponding to the metric $m(X, P)$. For any $P$ and $Q$ from $X$ the operators $\Delta(X, P)$ and $\Delta(X, Q)$ are isospectral and, therefore, the $\zeta$-regularized (modified, *i.e.*, with zero modes excluded) determinant $\det \Delta(X, P)$ is independent of $P \in X$ and, therefore, is

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a function on moduli space $\mathcal{M}_1$ of Riemann surfaces of genus one. The main result of the present work is the following explicit formula for this function:

$$\det\Delta(X,P) = C_1|\Im\sigma||\eta(\sigma)|^4F(t) = C_2\det\Delta^{(0)}(X)F(t), \tag{1.1}$$

where $\sigma$ is the $b$-period of the Riemann surface $X$, $C_1$ and $C_2$ are absolute constants, $\eta$ is the Dedekind eta-function, $\Delta^{(0)}$ is the Lapalacian on $X$ corresponding to the flat conformal metric of unit volume, the surface $X$ is represented as the two-fold covering of the Riemann sphere $\mathbb{C}P^1$ ramified over the points $0, 1, \infty$ and $t \in \mathbb{C} \setminus \{0,1\}$, and

$$F(t) = \frac{|t|^{\frac{1}{24}}|t-1|^{\frac{1}{24}}}{(|\sqrt{t}-1| + |\sqrt{t}+1|)^{\frac{1}{4}}}.$$

As is well known, the moduli space $\mathcal{M}_1$ coincides with the quotient space

$$\big(\mathbb{C} \setminus \{0,1\}\big)/G,$$

where $G$ is a finite group of order 6, generated by transformations $t \to \frac{1}{t}$ and $t \to 1-t$. A direct check shows that $F(t) = F(\frac{1}{t})$ and $F(t) = F(1-t)$, and, therefore, the right hand side of (1.1) is in fact a function on $\mathcal{M}_1$.

***Remark 1.1*** Using the classical relation (see, *e.g.,* [2, f-la (3.35)] )

$$t = -\Big(\frac{\Theta[^1_0](0\mid\sigma)}{\Theta[^0_1](0\mid\sigma)}\Big)^4,$$

one can rewrite the right-hand side as a function of $\sigma$ only.

The well known Ray–Singer relation $\det\Delta^{(0)} = C|\Im\sigma||\eta(\sigma)|^4$ (see [10–12]) used in (1.1) implies that (1.1) can be considered as a version of Polyakov's formula (relating determinants of the Laplacians corresponding to two smooth metrics in the same conformal class) for the case of two conformally equivalent metrics on a torus: one of them is smooth and flat, another is of curvature one and has exactly one singular point.

## 2 Metrics on the Base and on the Covering

Here we find an explicit expression for the metric $m$ on the Riemann sphere $\mathbb{C}P^1$ of curvature 1 and with three conical singularities at $P_1 = 0$, $P_2 = 1$, and $P_3 = \infty$.

The stereographic projection (from the south pole) maps the spherical triangle $T$ onto quarter of the unit disk $\{z \in \mathbb{C}\,;\, |z| \le 1,\ 0 \le \operatorname{Arg} z \le \pi/2\}$. The conformal map

$$z \longmapsto w = \Big(\frac{1+z^2}{1-z^2}\Big)^2 \tag{2.1}$$

sends this quarter of the disk to the upper half-plane $H$; the corner points $i, 0, 1$ go to the points $0, 1$, and $\infty$ on the real line. The push forward of the standard round metric

$$\frac{4|dz|^2}{(1+|z|^2)^2}$$

on the sphere by this map gives rise to the metric

$$m = \frac{|dw|^2}{|w||w-1|(|\sqrt{w}+1| + |\sqrt{w}-1|)^2} \tag{2.2}$$

on $H$; clearly, the latter metric can be extended (via the same formula) to $\mathbb{C}P^1$. The resulting curvature one metric on $\mathbb{C}P^1$ (also denoted by $m$) has three conical singularities of angle $\pi$: at $w = 0$, $w = 1$, and $w = \infty$.

Consider a two-fold covering of the Riemann sphere by a compact Riemann surface $X(t)$ of genus 1:

$$\mu\colon X(t) \to \mathbb{C}P^1 \tag{2.3}$$

ramified over four points: $0, 1, \infty$, and $t \in \mathbb{C} \smallsetminus \{0,1\}$. Clearly, the pull back metric $\mu^* m$ on $X(t)$ is a curvature one metric with exactly one conical singularity. The singularity is a conical point of angle $4\pi$ located at the point $\mu^{-1}(t)$.

## 3 Variation of Spectral Zeta-function with Respect to $t$

The analysis from [5] in particular implies that one can introduce the standard Ray–Singer $\zeta$-regularized determinant

$$\det \Delta^{\mu^* m} := \exp\{-\zeta'_{\Delta^{\mu^* m}}(0)\} \tag{3.1}$$

of the (Friedrichs) self-adjoint Laplacian $\Delta^{\mu^* m}$ in $L_2(X(t), \mu^* m)$, where $\zeta'_{\Delta^{\mu^* m}}$ is the spectral zeta-function. In this section we establish a formula for the variation of $\zeta'_{\Delta^{\mu^* m}}(0)$ with respect to the parameter $t$ (the fourth ramification point of the covering (2.3)). The derivation of this formula coincides almost verbatim with the proof of [5, Proposition 6.1]; therefore, we give only few details.

For the sake of brevity we identify the point $t$ of the base $\mathbb{C}P^1$ with its (unique) preimage $\mu^{-1}(t)$ on $X(t)$.

Let $Y(\lambda;\cdot)$ be the (unique) special solution of the Helmholz equation (here $\lambda$ is the spectral parameter) $(\Delta^m - \lambda)Y = 0$ on $X \smallsetminus \{t\}$ with asymptotic $Y(\lambda)(x) = \frac{1}{x} + O(x)$ as $x \to 0$, where $x(P) = \sqrt{\mu(P) - t}$ is *the distinguished holomorphic local parameter* in a vicinity of the ramification point $t \in X(t)$ of the covering (2.3). Introduce the complex-valued function $\lambda \mapsto b(\lambda)$ as the coefficient near $x$ in the asymptotic expansion

$$Y(x, \bar{x}; \lambda) = \frac{1}{x} + c(\lambda) + a(\lambda)\bar{x} + b(\lambda)x + O(|x|^{2-\epsilon}) \quad \text{as } x \to 0.$$

The following variational formula is proved in [5, Proposition 6.1]:

$$\partial_t\big(-\zeta'_{\Delta^{\mu^* m}}(0)\big) = \tfrac{1}{2}\big(b(0) - b(-\infty)\big). \tag{3.2}$$

The value $b(0)$ is found in [5, Lemma 4.2]: one has the relation

$$b(0) = -\tfrac{1}{6} S_{Sch}(x)\Big|_{x=0}, \tag{3.3}$$

where $S_{Sch}$ is the Schiffer projective connection on the Riemann surface $X(t)$.

Since $\lambda = -\infty$ is a local regime, in order to find $b(-\infty)$, the solution $Y$ can be replaced by a local solution with the same asymptotic as $x \to 0$. A local solution $\widehat{Y}$

with asymptotic

$$\widehat{Y}(u,\overline{u};\lambda) = \tfrac{1}{u} + \widehat{c}(\lambda) + \widehat{a}(\lambda)\overline{u} + \widehat{b}(\lambda)u + O(|u|^{2-\epsilon}) \quad \text{as } u \to 0$$

in the local parameter $u^2 = z - s$ was constructed in [5, Lemma 4.1] by separation of variables; here $z$ and $w = \mu(P)$ (resp. $s$ and $t$) are related by (2.1) (resp. by (2.1) with $z = s$ and $w = t$) and $\widehat{b}(-\infty) = \frac{1}{2}\frac{\overline{s}}{1+|s|^2}$. One can easily find the coefficients $A(t)$ and $B(t)$ of the Taylor series $u = A(t)x + B(t)x^3 + O(x^5)$. As a local solution replacing $Y$, we can take $A(t)\widehat{Y}$. This immediately implies that $b(-\infty) = A^2(t)\widehat{b}(-\infty) - B(t)/A(t)$. A straightforward calculation verifies that

$$b(-\infty) = \partial_t \log\Big(|t||t-1|(|\sqrt{t}+1| + |\sqrt{t}-1|)^2\Big)^{1/4}. \tag{3.4}$$

Observe that the right-hand side in (3.4) is actually the value of $\partial_w \log\rho(w,\overline{w})^{-1/4}$ at $w = t$, where $\rho(w,\overline{w})$ is the conformal factor of the metric (2.2); this is also a direct consequence of [4, Lemma 4].

Substituting (3.3) and (3.4) into (3.2), we obtain the desired formula for the variation of $\zeta'_{\Delta^{\mu^* m}}(0)$ with respect to the parameter $t$.

## 4 Explicit Formula for the Determinant

Equations (3.2), (3.3), and (3.4) imply that the determinant (3.1) can be represented as a product

$$\det \Delta^{\mu^* m} = C|\Im\sigma||\tau(t)|^2\Big|\frac{1}{|t||t-1|(|\sqrt{t}+1| + |\sqrt{t}-1|)^2}\Big|^{1/8}, \tag{4.1}$$

where $\tau(t)$ is the value of the Bergman tau-function (see [7–9]) on the Hurwitz space $H_{1,2}(2)$ of two-fold genus one coverings of the Riemann sphere, having $\infty$ as a ramification point at the covering, ramified over $1, 0, \infty$, and $t$. More specifically, $\tau$ is a solution of the equation

$$\partial_t \log\tau = -\tfrac{1}{12}S_B(x)|_{x=0},$$

where $S_B$ is the Bergman projective connection on the covering Riemann surface $X(t)$ of genus one and $x$ is the distinguished holomorphic parameter in a vicinity of the ramification point $t$ of $X(t)$. We remind the reader that the Bergman and the Schiffer projective connections are related via the equation

$$S_{Sch}(x) = S_B(x) - 6\pi(\Im\sigma)^{-1}v^2(x)$$

where $v$ is the normalized holomorphic differential on $X(t)$ and that the Rauch variational formula (see, *e.g.*, [7]) implies the relation

$$\partial_t \log\Im\sigma = \tfrac{\pi}{2}(\Im\sigma)^{-1}v^2(x)|_{x=0}.$$

The needed explicit expression for $\tau$ can be found *e.g.*, in [9, f-la (18)] (it is a very special case of the explicit formula for the Bergman tau-function on general coverings of arbitrary genus and degree found in [8] as well as of a much earlier formula of Kitaev and Korotkin for hyperelliptic coverings [6]). Namely, [9, f-la (18)] implies that

$$\tau = \eta^2(\sigma)\Big[\frac{v(\infty)^3}{v(P_1)v(P_2)v(Q)}\Big]^{\frac{1}{12}}, \tag{4.2}$$

where $P_1$ and $P_2$ are the points of the $X(t)$ lying over 0 and 1, $Q$ is a point of $X(t)$ lying over $t$ and $\infty$ denotes the point of the covering curve $X(t)$ lying over the point at infinity of the base $\mathbb{C}P^1$; $v$ is an arbitrary nonzero holomorphic differential on $X(t)$; and, say, $v(P_1)$ is the value of this differential in the distinguished holomorphic parameter at $P_1$. (One has to take into account that $\tau = \tau_I^{-2}$, where $\tau_I$ is from [9].) Taking

$$v = \frac{dw}{\sqrt{w(w-1)(w-t)}},$$

and using the following expressions for the distinguished local parameters at $P_1$, $P_2$, $Q$, and $\infty$

$$x = \sqrt{w}; \quad x = \sqrt{w-1}; \quad x = \sqrt{w-t}; \quad x = \frac{1}{\sqrt{w}}$$

one arrives at the relations (where ~ means = up to insignificant constants like ±2, etc.)

$$v(P_1) \sim \frac{1}{\sqrt{t}}; \quad v(P_2) \sim \frac{1}{\sqrt{t-1}}; \quad v(Q) \sim \frac{1}{\sqrt{t(t-1)}}; \quad v(\infty) \sim 1.$$

These relations together with (4.2) and (4.1) imply (1.1).

***Remark 4.1*** The result of this paper can be generalized to hyperelliptic surfaces of genus $g \geq 2$. Indeed, choose $2g-1$ distinct points $Q_1, Q_2, \ldots, Q_{2g-1}$ in $\mathbb{C}P^1 \setminus \{P_1, P_2, P_3\}$ and consider the two-fold covering

$$\mu_g \colon X(Q_1, Q_2, \ldots, Q_{2g-1}) \to \mathbb{C}P^1$$

ramified over $Q_1, \ldots, Q_{2g-1}$ and $P_1, P_2, P_3$. The pullback $\mu_g^* m$ of the metric $m$ in (2.2) by $\mu_g$ is a metric of constant curvature 1 with conical points of angle $4\pi$ at $2g-1$ Weierstrass points of the hyperelliptic curve $X(Q_1, Q_2, \ldots, Q_{2g-1})$ (three remaining Weierstrass points are nonsingular points of the metric). Using the same methods as in the genus 1 case, one can derive an explicit expression for the determinant of the Laplacian in the metric $\mu_g^* m$ as a function on moduli space of hyperelliptic curves of genus $g$. For instance, in genus two one gets the following explicit expression

$$\det \Delta^{\mu_2^* m} = C \mathcal{F}^{2/5} \Phi(t_1, t_2, t_3),$$

where

$$\mathcal{F} = (\det \Im \mathbb{B})^{5/2} \prod_{\beta} |\Theta[\beta](0|\mathbb{B})|$$

is the Petersson norm $\|\Delta_2\|$ of the Siegel cusp form $\Delta_2 = \prod_\beta \Theta[\beta](0|\mathbb{B})$ ($\beta$ runs through the set of 10 even characteristics) and

$$\Phi(t_1, t_2, t_3) = \frac{|t_1 t_2 t_3 (t_1-1)(t_2-1)(t_3-1)|^{-\frac{1}{40}} |t_1-t_2|^{\frac{1}{10}} |t_1-t_3|^{\frac{1}{10}} |t_2-t_3|^{\frac{1}{10}}}{\prod_{k=1}^{3} (|\sqrt{t_k}+1| + |\sqrt{t_k}-1|)^{\frac{1}{4}}},$$

where the points $Q_1, Q_2, Q_3, P_1, P_2, P_3$ are identified with the points $t_1, t_2, t_3, 0, 1, \infty$ of $\mathbb{C}P^1$. It is straightforward to check that the right-hand side of (4.1) is a function on the moduli space $\mathcal{M}_2$ of compact Riemann surfaces of genus 2 (it suffices to check that $\Phi(t_1, t_2, t_3) = \Phi(t_1^{-1}, t_2^{-1}, t_3^{-1}) = \Phi(1-t_1, 1-t_2, 1-t_3)$).

***Remark 4.2*** [In response to referee comments] The necessary and sufficient condition on a triple of positive numbers $\theta_1$, $\theta_2$, $\theta_3$ for the existence of a conformal curvature one metric on the Riemann sphere $\mathbb{C}P^1$, with three conic singularities of angles $2\pi\theta_1$, $2\pi\theta_2$, $2\pi\theta_3$ at the points 0, 1, and $\infty$, respectively, was obtained in [3, 13]. Let $m = \rho(w, \overline{w})|dw|^2$ stand for the corresponding metric on $\mathbb{C}P^1$. Then the pull back metric $\mu^* m$ on $X(t)$ (here $\mu$ is the same as in (2.3)) is a curvature one metric with conical singularity of angle $4\pi$ located at the point $\mu^{-1}(t)$ and three conical singularities of angles $4\pi\theta_1, 4\pi\theta_2, 4\pi\theta_3$ at the points $\mu^{-1}(0)$, $\mu^{-1}(1)$, and $\mu^{-1}(\infty)$, respectively. It turns out that the formula (3.2) (for the spectral zeta function of the Friedrichs self-adjoint extension of Laplacian $\Delta^{\mu^* m}$) is still valid, where $b(0)$ is the same as before and $b(-\infty) = \partial_w \log \rho(w, \overline{w})^{-1/4}|_{w=t}$. For details, we refer the reader to [4]. As a generalization of (1.1), we thus obtain

$$\begin{aligned} \det \Delta^{\mu^* m} &= C_1 \Im\sigma \|\eta(\sigma)|^4 \sqrt[12]{|t^2 - t|} \sqrt[8]{\rho(t, \bar{t})} \\ &= C_2 \det \Delta^{(0)}(X) \sqrt[12]{|t^2 - t|} \sqrt[8]{\rho(t, \bar{t})}, \end{aligned} \tag{4.3}$$

where $C_1$ and $C_2$ are absolute constants and $t$ can be expressed as a function of $\sigma$; see Remark 1.1. Having at hand an explicit expression for the conformal factor $\rho(w, \overline{w})$ (in the case $\theta_1 = \theta_2 = \theta_3 = 1/2$ we use (2.2)), one immediately gets the corresponding explicit formula for $\det \Delta^{\mu^* m}$. Let us also note that (4.3) remains valid if $m = \rho(w, \overline{w})|dw|^2$ is any conical metric on $\mathbb{C}P^1$ and $t$ stays outside of the conical singularities of $m$.

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