Алгебра и анализ
Том. 16 (2004), вып. 3

St. Petersburg Math. J.
Vol. 16 (2005), No. 3, Pages 477–506
S 1061-0022(05)00862-9
Article electronically published on May 2, 2005

ON THE ASYMPTOTICS OF SOLUTIONS TO THE NEUMANN PROBLEM FOR HYPERBOLIC SYSTEMS IN DOMAINS WITH CONICAL POINTS

A. KOKOTOV AND B. PLAMENEVSKIĬ

Abstract. Hyperbolic systems of second-order differential equations are considered in a domain with conical points at the boundary; in particular, the equations of elastodynamics are discussed. The asymptotics of solutions near conical points is studied. The "hyperbolic character" of the asymptotics shows itself in the properties of the coefficients (stress intensity factors) depending on time. Some formulas for the coefficients are presented and sharp estimates in Soboloev's norms are proved.

§1. Introduction

Let G be a domain in $\mathbb{R}^n$ with boundary ∂G containing conical points. We consider a class of hyperbolic systems of second-order differential equations with Neumann's boundary conditions in the cylinder $G \times \mathbb{R} = \{(x,t) : x \in G, t \in \mathbb{R}\}$ (and in the semicylinder $G \times \mathbb{R}_+$). In particular, this class includes the dynamical equations of elasticity theory. Our main purpose is to study the asymptotics of solutions near the conical points. We investigate the solvability of the problem mentioned above in a scale of weighted spaces. This enables us to obtain and justify asymptotic formulas. For the coefficients in the asymptotics (depending on time), we give explicit formulas and sharp estimates in Sobolev's norms.

The principal part of the asymptotics near a conical point is a linear combination $\sum c_j(t)u_j(x)$ of functions u_j satisfying a homogeneous elliptic problem in the "tangent" cone; the latter problem is the elliptic part of the initial problem. The hyperbolic character of the asymptotics shows itself in the coefficients c_j. They admit representations of the form

$$c_j(t) = \int_G \int_{-\infty}^{+\infty} f(x, t-s)\overline{w_j(x,s)}\, dx\, ds, \tag{1.1}$$

where f is the right-hand side of the hyperbolic system in question (we consider the homogeneous boundary conditions), and the w_j are some functions satisfying the homogeneous problem in the cylinder $G \times \mathbb{R}$ and determined by their asymptotics near the conical point. The proof of the above formulas is the main result of the paper.

The properties of coefficients are of special interest for elastodynamics (the stress intensity factors). In the theory of elliptic boundary-value problems, the corresponding formulas for the coefficients made it possible to study the asymptotics of fundamental solutions (the Green functions and the Poisson kernels) near conical points ([26]; see also [7]). It can be expected that formulas (1.1) will play a similar role for the fundamental solutions to hyperbolic problems.

1991 *Mathematics Subject Classification.* Primary 35C20, 35L20.
Key words and phrases. Hyperbolic systems, weighted estimates, asymptotics.

The present paper contains no further investigation of the functions w_j. We calculate these functions explicitly for the wave equation in a cone. The expressions obtained show that $w_j(x,t) = 0$ for $t < |x|$, and the singular support satisfies $\operatorname{sing\,supp} w_j \subset \{(x,t) : |x| = t\}$. This and (1.1) directly yield some consequences for the coefficients c_j. (For instance, if the singular support of f is bounded in the spatial variables and upper bounded in time, then a "back edge" phenomenon occurs for the coefficients in the asymptotics: the functions c_j become smooth after the moment when the perturbation coming from the singular support of f leaves the vertex of the cone.)

The study of the model problem in an n-dimensional cone K constitutes the bulk of the paper. As in [5, 6], the method is based on "combined" *a priori* estimates of solutions. The Fourier transformation relative to time leads to a problem in the cone K with parameter $\tau = \sigma - i\gamma$, where $\sigma \in \mathbb{R}$ and $\gamma > 0$. In a neighborhood of the vertex (of diameter $\mathrm{const}/|\tau|$) we employ a weighted elliptic estimate. Localizing a global energy estimate, we obtain a weighted hyperbolic inequality far from the vertex. Requiring some additional smoothness of the data with respect to time, we match the hyperbolic inequality with the elliptic estimate in the intermediate zone.

As has already been mentioned, we consider scales of weighted spaces. Roughly speaking, as the weight we take $|x|^\beta \exp(-\gamma t)$, where $|x|$ is the distance from the vertex of K. The combined estimates are proved for all $\beta \le 1$, $\beta \notin \{\beta_k\}$, where $\{\beta_k\}$ is a sequence such that $1 > \beta_1 > \beta_2, \dots, \beta_k \to -\infty$. In the cone, the problem involving a parameter gives rise to a closed operator. The kernel and cokernel of this operator are trivial if $\beta \in (\beta_1, 1]$. As β decreases, the dimension of the cokernel increases (when β crosses β_k) but remains finite. The elements of a basis of the cokernel are uniquely determined by their asymptotics near the vertex. This allows us to obtain the asymptotics of solutions near the vertex of the cone, including formulas for the coefficients. The inverse Fourier transformation carries the theory over to the problem with time, posed in the cone.

To implement the above method for the Neumann problem, we need to modify the argument used in [6] in the case of the Dirichlet boundary condition. In particular, the localization procedure becomes more complicated. The proof of a global weighted estimate for solutions of the problem with parameter in a cone becomes more involved, as well as the estimate itself. (Technically, these complications are due to the fact that multiplication by a cut-off function and the operator of boundary condition do not commute in general.) Dealing with the equations of elasticity theory, we employ a nonstandard "Korn inequality" proved in [2]. The corresponding weighted spaces must be defined in a special way for $n = 2$: a weaker "nonhomogeneous" norm must be employed to incorporate the generalized (strong) solutions in a proper weighted space (this is well known in the theory of elliptic boundary-value problems in domains with piecewise smooth boundaries; see, e.g., [12, 7]; here we merely adapt the corresponding techniques to our problem).

The paper consists of seven sections. A strong solution for the problem with parameter in a cone is introduced in §2; the combined estimates of solutions are established in §3. In §4 we study the properties of the operator of this problem. The asymptotics of solutions of the problem with time in a cone is described in §5. These results are specified for the wave equation in §6. Finally, in §7 we explain briefly how these results can be extended to the problem in the cylinder $G \times \mathbb{R}$.

Let us indicate some publications related to hyperbolic problems in nonsmooth domains; however, the results of that work are not used here. The wave equation was considered in [13] (a wedge with edge of codimension 2, an explicit formula for solutions), in [14] (domains with edges, the fundamental solution, propagation of singularities; the approach was based on a functional calculus for the Laplace operator), in [15, 16] (microlocal analysis), and in [17] (the homogeneous Dirichlet boundary condition); see also the references in these papers. The monographs [18]–[20] were devoted to the case of rectilinear boundary and were mostly built on explicit formulas. A certain general approach different from that presented in [5] and [6] was proposed in [21, 22] (strongly hyperbolic systems in a domain with a conical point, the homogeneous Dirichlet boundary conditions); this approach does not lead to formulas for the coefficients.

§2. The model problems in a cone. A strong solution

2.1. The problem in a cone. Let K be an open cone in $\mathbb{R}^n$ with boundary ∂K and with vertex O at the origin. Let Ω be the (relatively) open set cut out by K from the $(n-1)$-dimensional unit sphere S^{n-1}. We assume that the boundary $\partial\Omega$ is smooth. We introduce a hyperbolic operator $L(D_x, D_t) = P(D_x) - D_t^2$ with elliptic part $P(D_x) = \sum A_{pq} D_{x_p} D_{x_q}$, where $1 \le p, q \le n$, the A_{pq} are $(m\times m)$-matrices with constant complex entries, and $A_{qp} = A_{pq}^*$. The operator $P(D_x)$ is subject to one of the following two conditions:

a)$\sum\langle A_{pq}\eta_p, \eta_q\rangle \ge c|\eta|^2$ for all $\eta_p \in \mathbb{C}^m$, where $c > 0$, $|\eta|^2 = \sum |\eta_p|^2$, and $\langle\cdot,\cdot\rangle$ denotes the inner product in $\mathbb{C}^m$;

b)$P(D_x)$ coincides with the Lamé operator $-\mu\Delta - (\lambda+\mu)\operatorname{grad}\operatorname{div}$ (and then $m = n = 2, 3$).

In the cylinder $Q = \{(x,t) : x \in K, t \in \mathbb{R}\}$, consider the problem

$$\begin{cases} L(D_x, D_t)u(x,t) = f(x,t), & (x,t) \in Q, \\ N(x, D_x)u(x,t) = 0, & (x,t) \in \partial Q. \end{cases} \tag{2.1}$$

Here $N(x, D_x)$ is the $(m \times m)$-matrix of first-order differential operators in the Green formula

$$(P(D_x)u, v)_K = a(u, v; K) - (Nu, v)_{\partial K}, \quad u, v \in C_c^\infty(\overline{K}), \tag{2.2}$$

and $a(\cdot,\cdot;\mathcal{W})$ is a symmetric sesquilinear form (i.e., $a(u,v;K) = \overline{a(v,u;\mathcal{W})}$), specifically,

$$a(u, v; K) = \int_{\mathcal{W}} \alpha(\nabla u, \nabla v)\, dx = \int_K \sum_{r,s=1}^{n} \sum_{\beta,\gamma=1}^{m} a_{\beta\gamma rs} \partial_{x_r} u_\beta \overline{\partial_{x_s} v_\gamma}\, dx. \tag{2.3}$$

We assume that the operator $\{P(D_x), N(x, D_x)\}$ is elliptic. In particular, if $P(D_x) = -\mu\Delta - (\lambda+\mu)\operatorname{grad}\operatorname{div}$ with $\lambda \ge 0$ and $\mu > 0$, then we put $Nu = \{\sum_k \sigma_{jk}(u)\nu_k\}_{j=1}^3$, where $\{\sigma_{jk}\}$ is the stress tensor,

$$\sigma_{jk}(u) = \mu\left(\frac{\partial u_j}{\partial x_k} + \frac{\partial u_k}{\partial x_j}\right) + \delta_{jk}\lambda \operatorname{div} u,$$

and $\nu = \{\nu_1, \nu_2, \nu_3\}$ is the outward normal to the boundary.

Setting $\eta_p = \xi_p\zeta$ with $\xi_p \in \mathbb{C}$ and $\zeta \in \mathbb{C}^m$ in condition a), we obtain

$$\sum_{1\le s,p\le n} (A_{sp}\zeta, \zeta)\xi_s\xi_p \ge c|\xi|^2|\zeta|^2,$$

which is equivalent to the strong ellipticity of the operator $P(D_x)$. However, for the Lamé operator (which is known to be strongly elliptic) condition a) fails; the energy form $\alpha(\cdot,\cdot)$ is no better than nonnegative.

As an example, we also consider the Neumann problem for the scalar wave equation

$$(\partial_t^2 - \Delta)u = f \text{ on } Q, \quad \partial_\nu u = 0 \text{ on } \partial Q; \tag{2.4}$$

in this case some of our results become more explicit and their proofs shorten.

2.2. Energy estimates on solutions of the problem with parameter. Applying the Fourier transformation $F_{t\to\tau}$ to equations (2.1), we obtain a problem with parameter τ:

$$\begin{cases} L(D_x,\tau)\hat{u}(x,\tau) = \hat{f}(x,\tau), & x \in K, \\ N(x,D_x)\hat{u}(x,\tau) = 0, & x \in \partial K. \end{cases} \tag{2.5}$$

Proposition 2.1. *Suppose $v \in C_c^\infty(\bar{K})$ and $N(x,D_x)v = 0$ on ∂K. Then the estimate*

$$\gamma^2 \int_K (p^2|v(x)|^2 + |\nabla v(x)|^2)\,dx \le c \int_K |L(D_x,\sigma - i\gamma)v(x)|^2\,dx \tag{2.6}$$

is true with a constant c independent of the parameter $\tau = \sigma - i\gamma$, where $\sigma \in \mathbb{R}$, $\gamma > 0$, and $p = |\tau|$.

Let $H^1(K)$ denote the usual Sobolev space in K. We preface the proof of the proposition with the following assertion.

Lemma 2.2. *Let $a(\cdot,\cdot;K)$ be the same form as in (2.2). Then for all $u \in H^1(K)$ we have*

$$\|\nabla u; L_2(K)\|^2 \le c\, a(u,u;K), \tag{2.7}$$

where c is a constant independent of u.

Proof of the lemma. If $P(D_x)$ satisfies condition a), then for the form $\alpha(\cdot,\cdot)$ in (2.3) we have

$$\alpha(\nabla u, \nabla u) = \sum (A_{pq}\partial_{x_p}u, \partial_{x_q}u) \ge c|\nabla u|^2,$$

which leads to (2.7). If $P(D_x)$ is the Lamé operator, then

$$a(u,u;K) \ge E(u;K), \tag{2.8}$$

where

$$E(u;K) = \int_K \sum_{j,h} |\partial_{x_h}u_j + \partial_{x_j}u_h|^2\,dx$$

(see, e.g., [3]). In this case, (2.7) follows from (2.8) and "the Korn inequality"

$$E(u;\mathcal{K}) \ge c\|\nabla u; L_2(\mathcal{K})\|^2, \quad u \in H^1(K), \;\; c > 0. \tag{2.9}$$

It should be emphasized that inequality (2.9) is "nonclassical" (the domain K is unbounded and there is no summand $\|u; L_2(K)\|^2$ on the left in (2.9)). Estimate (2.9) is contained in [2, Theorem 3.1]. $\square$

Proof of Proposition 2.1. Put

$$f(x,t) = \partial_t^2 w(x,t) + P(D_x)w(x,t)$$

with w lying in the Schwartz space $S(\mathbb{R}^{n+1}_{x,t})$ and satisfying $N(x,D_x)w = 0$ on ∂K. We have

$$\int_K \int_{-\infty}^t (\langle w_{tt}, w_t\rangle + \langle P(D_x)w, w_t\rangle)\,dx\,dt = \int_K \int_{-\infty}^t \langle f, w_t\rangle\,dx\,dt.$$

Adding this identity to its complex conjugate, we obtain

$$\begin{aligned} &\int_K \int_{-\infty}^t (\partial_t|w_t|^2 + \langle P(D_x)w, w_t\rangle + \langle w_t, P(D_x)w\rangle)\,dx\,dt \\ &\qquad = 2\Re \int_K \int_{-\infty}^t \langle f, w_t\rangle\,dy\,dt. \end{aligned} \tag{2.10}$$

Formula (2.2) yields

$$\begin{aligned}&(P(D_x)w, w_t)_K + (w_t, P(D_x)w)_K\\ &\quad = (P(D_x)w, w_t)_K + (P(D_x)w_t, w)_K = \partial_t (P(D_x)w, w)_K.\end{aligned}$$

Therefore, (2.10) can be rewritten as

$$\|w_t(\cdot,t); L_2(K)\|^2 + (P(D_x)w(\cdot,t), w(\cdot,t))_K = 2\Re \int_K \int_{-\infty}^{t} \langle f, w_t\rangle\, dx\, dt.$$

Using (2.2), (2.7), and the relation $N(x, D_x)w|\partial K = 0$, we arrive at the inequality

$$\begin{aligned}&\|w_t(\cdot,t); L_2(K)\|^2 + \|\nabla_x w(\cdot,t); L_2(K)\|^2\\ &\qquad \le c\int_{-\infty}^{t} \|f(\cdot,t); L_2(K)\|\, \|w_t(\cdot,t); L_2(K)\|\, dt.\end{aligned} \tag{2.11}$$

We denote by $h(t)$ the integrand on the right, multiply (2.11) by $e^{-2\gamma t}$, and integrate to obtain

$$\begin{aligned}&\int_{-\infty}^{+\infty} e^{-2\gamma s}(\|w_s(\cdot,s); L_2(K)\|^2 + \|\nabla_x w(\cdot,s); L_2(K)\|^2)\\ &\quad \le c\int_{-\infty}^{+\infty} e^{-2\gamma s}\int_{-\infty}^{s} h(r)\, dr = c\int_{-\infty}^{+\infty} h(r)\, dr \int_r^{+\infty} e^{-2\gamma s}\, ds\\ &\quad \le (c\gamma^{-1}/2)\int_{-\infty}^{+\infty} h(r) e^{-2\gamma r}\, dr\\ &\quad \le (c\gamma^{-1}/2)\Big(\int_{-\infty}^{+\infty} e^{-2\gamma t}\|f(\cdot,t)\|^2\, dt\Big)^{1/2}\Big(\int_{-\infty}^{+\infty} e^{-2\gamma t}\|w_t(\cdot,t)\|^2\, dt\Big)^{1/2}.\end{aligned}$$

Consequently,

$$\begin{aligned}&\gamma^2\int_{-\infty}^{+\infty} e^{-2\gamma t}(\|w_t(\cdot,t); L_2(K)\|^2 + \|\nabla_x w(\cdot,t); L_2(K)\|^2)\, dt\\ &\qquad \le c\int_{-\infty}^{+\infty} e^{-2\gamma t}\|f(\cdot,t)\|^2\, dt.\end{aligned} \tag{2.12}$$

In (2.12) we put $w(x,t) = v(x)\psi(t)$, where $\psi \in e^{-\gamma t}S(\mathbb{R}) \cap S(\mathbb{R})$, $v \in C_c^\infty(\bar K)$, and $N(x, D_x)v|\partial K = 0$. We have

$$\begin{aligned}&\gamma^2\int_K dx \int_{-\infty}^{+\infty} d\sigma |\hat\psi(\sigma - i\gamma)|^2 (p^2|v(x)|^2 + |\nabla_x v(x)|^2)\\ &\qquad \le c\int_K dx\int_{-\infty}^{+\infty} d\sigma |\hat\psi(\sigma - i\gamma)|^2 |L(D_x, \sigma - i\gamma)v(x)|^2.\end{aligned}$$

Since the function $\psi \in e^{-\gamma t}S(\mathbb{R}) \cap S(\mathbb{R})$ is arbitrary, this justifies (2.6). □

2.3. A weak solution of the problem with parameter. To simplify the notation, we rewrite (2.5) in the form

$$L(D_x, \tau)u = \{-\tau^2 + P(D_x)\}u = f \text{ in } K; \quad N(x, D_x)u = 0 \text{ on } \partial K. \tag{2.13}$$

As usual, a function $u \in H^1(K)$ is called a *weak solution* of problem (2.13) (with $f \in L_2(K)$) if

$$B(u,v) := a(u,v;K) - \tau^2\int_K u\bar v\, dx = \int_K f\bar v\, dx \tag{2.14}$$

for any v in $H^1(K)$. The existence of a unique weak solution of (2.13) can be obtained easily from the Lax–Milgram–Vishik lemma (in the form given in [1, Remark 2.9.3]) combined with the following assertion.

Proposition 2.3. *If* $\tau^2 \in \mathbb{C} \setminus \bar{\mathbb{R}}_+$, *then*

$$|B(u,u)| \geq \delta \|u; H^1(K)\|^2$$

with $\delta = \delta(\tau) > 0$.

Proof. Let $\alpha = \|u; L_2(K)\|^2$, and let $\beta = a(u,u;K)$. By (2.7), we have

$$\begin{aligned}(\alpha^2+\beta^2)^{1/2} &\geq 2^{-1/2}(\alpha+\beta)\\ &\geq 2^{-1/2}(\|u; L_2(K)\|^2 + c_1(\|\nabla u; L_2(K)\|^2) \geq c_2\|u; H^1(K)\|^2.\end{aligned}$$

Therefore, it suffices to show that

$$|B(u,u)|^2 \geq \delta^2(\alpha^2+\beta^2). \tag{2.15}$$

Setting $\tau^2 = \sigma_1 + i\gamma_1$ with real σ_1 and γ_1, we obtain $|B(u,u)|^2 = \beta^2 + (\sigma_1^2+\gamma_1^2)\alpha^2 - 2\sigma_1\alpha\beta$. If $\sigma_1 = 0$ and $\gamma_1 \neq 0$ or $\sigma_1 < 0$, then (2.15) is obvious. If $\sigma_1 > 0$, we have $\gamma_1 \neq 0$. Choose $\varepsilon \in \mathbb{R}$ such that $\varepsilon^2 < 1$ and $(\varepsilon^{-2}-1)\sigma_1^2 < \gamma_1^2$. Then $2\sigma_1\alpha\beta \leq \varepsilon^2\beta^2 + \varepsilon^{-2}\sigma_1^2\alpha^2$ and

$$|B(u,u)|^2 \geq \beta^2(1-\varepsilon^2) + \alpha^2[\gamma_1^2 - (\varepsilon^{-2}-1)\sigma_1^2] \geq \delta^2(\alpha^2+\beta^2). \qquad \square$$

2.4. A strong solution of the problem with parameter. We view the map $A(\tau)$: $u \mapsto L(D_x,\tau)u$ as an unbounded operator in $L_2(K)$ with domain

$$\{u \in C^\infty(\bar{K} \setminus O) \cap H^1(K) : N(x,D_x)u|\partial K = 0, L(D_x,\tau)u \in L_2(K)\}.$$

This operator admits closure. From now on, $A(\tau)$ stands for this closure with domain $D(A(\tau))$.

Definition 2.4. Let $f \in L_2(K)$. A solution of the equation $A(\tau)u = f$ is called a *strong solution* of problem (2.13).

Theorem 2.5. *For any* $f \in L_2(K)$ *there exists a unique strong solution* u *of problem* (2.13). *The estimate*

$$\gamma^2(p^2\|u; L_2(K)\|^2 + \|\nabla u; L_2(K)\|^2) \leq c\|f; L_2(K)\|^2 \tag{2.16}$$

is fulfilled with a constant c *independent of* $\tau = \sigma - i\gamma$, *where* $\sigma \in \mathbb{R}$, $\gamma > 0$, *and* $p = |\tau|$.

Proof. Estimate (2.6) remains valid for $v \in D(A(\tau))$. Hence, the kernel of $A(\tau)$ is trivial and the range of the operator is closed. Moreover, $C_c^\infty(K) \subset \operatorname{im} A(\tau)$, because the weak solution of problem (2.13) with $f \in C_c^\infty(K)$ belongs to $D(A(\tau))$ in accordance with the known results of the theory of elliptic boundary-value problems. Since $\operatorname{im} A(\tau)$ is closed, this implies that $\operatorname{im} A(\tau) = L_2(K)$. $\square$

§3. Weighted estimates of solutions of the problem with parameter in a cone

3.1. Estimates far from the vertex of the cone. Let s be a nonnegative integer, and let $\beta \in \mathbb{R}$. We introduce the space $H^s_\beta(K)$ as the completion of $C_c^\infty(\overline{K} \setminus O)$ with respect to the norm

$$\|u; H^s_\beta(K)\| = \left(\sum_{|\alpha| \leq s} \int_K |y|^{2(\beta - s + |\alpha|)} |D_y^\alpha u(y)|^2\, dy\right)^{1/2}. \tag{3.1}$$

The space $H^s_\beta(K;q)$ with positive parameter q is equipped with the norm

$$\|u; H^s_\beta(K;q)\| = \left(\sum_{k=0}^{s} q^{2k} \|u; H^{s-k}_\beta(K)\|^2 \right)^{1/2}. \tag{3.2}$$

We also put

$$\|u; H^0_\beta(\partial K)\| = \left(\int_{\partial K} |u(x)|^2 |x|^{2\beta} \, dx \right)^{1/2}.$$

Proposition 3.1. *Let $\kappa_\infty \in C^\infty(\mathbb{R}^n)$ be a cut-off function such that $0 \le \kappa_\infty \le 1$, $\kappa_\infty(\eta) = 1$ for $|\eta| > c_1 > 0$, and $\kappa_\infty(\eta) = 0$ for $|\eta| < c_1/2$. Let $\Psi \in C^\infty(\mathbb{R}^n)$ be another cut-off function such that $0 \le \Psi \le 1$, $\Psi(\eta) = 1$ for $c_1/2 < |\eta| < C_1 p^2/2\gamma^2$, and $\Psi(\eta) = 0$ for $|\eta| < c_1/4$ and $|\eta| > C_1 p^2/\gamma^2$. Then the constant C_1 can be chosen so that the estimate*

$$\begin{aligned} &(\gamma/p)^2 \|\kappa_\infty u; H^1_\beta(K;1)\|^2 \\ &\quad \le c\{\|\kappa_\infty L(D_\eta, \theta) u; H^0_\beta(K)\|^2 + \|\Psi u; H^1_{\beta-1}(K;1)\|^2 + \|\Psi u; H^0_{\beta-1/2}(\partial K)\|^2\} \end{aligned} \tag{3.3}$$

is valid for all $u \in C^\infty_c(\bar{K} \setminus O)$ satisfying $N(\eta, D_\eta, \theta) u | \partial K = 0$; here $\theta = \tau/p$ with $p = |\tau|$. The constants c, c_1, and C_1 are independent of $\tau = \sigma - i\gamma$, where $\sigma \in \mathbb{R}$ and $\gamma > 0$.

Proof. We choose κ and ψ in $C^\infty_c(\mathbb{R}^n)$ with $\operatorname{supp} \kappa \subset \{1/2 < |y| < 2\}$, $\operatorname{supp} \psi \subset \{1/4 < |y| < 4\}$, and $\kappa\psi = \kappa$. Note that if $u \in C^\infty_c(\bar{K} \setminus O)$ and $N(x, D_x) u | \partial K = 0$, then

$$\|N(x, D_x)(\kappa u); \partial K\| = \|[N(x, D_x), \kappa] u; \partial K\| \le C \|\psi u; \partial K\|; \tag{3.4}$$

here and in what follows we denote $\|w; M\| := \|w; L_2(M)\|$. We assume that $w \in C^\infty_c(\bar{K} \times \mathbb{R})$ and $N(x, D_x) w | \partial K = 0$, and put

$$f_\kappa(x,t) = (\partial_t^2 + P(D_x))(\kappa w(x,t)).$$

Arguing in the same way as in (2.10), we obtain

$$\begin{aligned} &\int_K \int_{-\infty}^{t} (\partial_t |\kappa w_t|^2 + \langle P(D_x)(\kappa w), \kappa w_t \rangle + \langle \kappa w_t, P(D_x)(\kappa w) \rangle) \, dx \, dt \\ &\qquad = 2\Re \int_K \int_{-\infty}^{t} \langle f_\kappa, \kappa w_t \rangle \, dx \, dt. \end{aligned} \tag{3.5}$$

Formulas (2.2) and (3.4) lead to the relations

$$\begin{aligned} &(P(D_x)(\kappa w), \kappa w_t)_K + (\kappa w_t, P(D_x)(\kappa w))_K \\ &\quad = (\kappa w, P(D_x)(\kappa w_t))_K + (\kappa w_t, P(D_x)(\kappa w))_K - (N(x, D_x)(\kappa w), \kappa w_t)_{\partial K} \\ &\qquad + (\kappa w, N(x, D_x)(\kappa w_t))_{\partial K} \\ &\quad = \partial_t \Re (P(D_x)(\kappa w), \kappa w)_K - \Re\{(N(x, D_x)(\kappa w), \kappa w_t)_{\partial K} - (\kappa w, N(x, D_x)(\kappa w_t))_{\partial K}\} \\ &\quad \ge \partial_t \Re (\kappa w, P(D_x)(\kappa w))_K - c \|\psi w; \partial K\| \, \|\psi w_t; \partial K\|. \end{aligned}$$

Together with (3.5), this gives

$$\begin{aligned} &\|\kappa w_t(\cdot, t); K\|^2 + \Re (P(D_x)(\kappa w(\cdot, t)), \kappa w(\cdot, t))_K \\ &\qquad \le c_2 \left(\int_{-\infty}^{t} \|\psi w; \partial K\| \, \|\psi w_t; \partial K\| \, dt + \int_{-\infty}^{t} \|f_\kappa; K\| \, \|\kappa w_t; K\| \, dt \right). \end{aligned} \tag{3.6}$$

On the other hand,

$$\begin{aligned} \Re (P(D_x)(\kappa w), \kappa w)_K &= a(\kappa w, \kappa w; K) - \Re (N(x, D_x)(\kappa w), \kappa w)_{\partial K} \\ &\ge c_3 \|\nabla_x (\kappa w(\cdot, t)); K\|^2 - c_4 \|\psi w; \partial K\|^2. \end{aligned} \tag{3.7}$$

Comparison of (3.6) and (3.7) yields
(3.8)
$$\|\kappa w_t(\cdot,t);K\|^2+\|\nabla_x(\kappa w(\cdot,t));K\|^2 \\ \leq c_5\Big(\int_{-\infty}^t\|\psi w;\partial K\|\,\|\psi w_t;\partial K\|\,dt+\int_{-\infty}^t\|f_\kappa;K\|\,\|\kappa w_t;K\|\,dt+\|\psi w;\partial K\|^2\Big).$$

We multiply (3.8) by $e^{-2\gamma t}$ and integrate. As in the proof of (2.12), we obtain the inequality

$$\int_{-\infty}^{+\infty}e^{-2\gamma t}\big\{\|\kappa w_t(\cdot,t);K\|^2+\|\nabla_x(\kappa w(\cdot,t));K\|^2\big\}\,dt \\ \leq (c_6/\gamma)\Big(\int_{-\infty}^{+\infty}e^{-2\gamma t}(\|\psi w;\partial K\|\,\|\psi w_t;\partial K\|+\|f_\kappa;K\|\,\|\kappa w_t;K\|)\,dt \\ +c_7\int_{-\infty}^{+\infty}e^{-2\gamma t}\|\psi w;\partial K\|^2\,dt\Big).$$

Since

$$\gamma^{-1}\|\psi w\|\,\|\psi w_t\|\leq 2^{-1}(\|\psi w\|^2+\gamma^{-2}\|\psi w_t\|)$$

and

$$\gamma^{-1}\|f_\kappa\|\,\|\kappa w_t\|\leq C_\varepsilon\gamma^{-2}\|f_\kappa\|^2+\varepsilon\|\kappa w_t\|^2,$$

we have

$$\gamma^2\int_{-\infty}^{+\infty}e^{-2\gamma t}\{\|\kappa w_t(\cdot,t);K\|^2+\|\nabla_x(\kappa w(\cdot,t));K\|^2\}\,dt \\ \leq c\int_{-\infty}^{+\infty}e^{-2\gamma t}(\|f_\kappa;K\|^2+\gamma^2\|\psi w;\partial K\|^2+\|\psi w_t;\partial K\|^2)\,dt. \tag{3.9}$$

In (3.9) we take $w(x,t)=v(x)\chi(t)$, where $v\in C_c^\infty(\bar K)$, $N(x,D_x)v=0$ on ∂K, and $\chi\in e^{-\gamma t}S(\mathbb{R})\cap S(\mathbb{R})$. Then

$$\gamma^2(|\tau|^2\|\kappa v;K\|^2+\|\nabla_x(\kappa v);K\|^2) \\ \leq c(\|L(D_x,\tau)(\kappa v);K\|^2+\gamma^2\|\psi v;\partial K\|^2+|\tau|^2\|\psi v;\partial K\|^2). \tag{3.10}$$

Observe that

$$\|L(D_x,\tau)(\kappa v);K\|\leq\|\kappa L(D_x,\tau)v;K\|+\|[L(D_x,\tau),\kappa]v;K\| \\ \leq\|\kappa L(D_x,\tau)v;K\|+c(p\|\psi v;K\|+\|\psi v;K\|+\|\nabla(\psi v);K\|). \tag{3.11}$$

Using the inequalities $\|\psi v;K\|\leq c\|\nabla(\psi v);K\|$, (3.10), and (3.11), we arrive at the estimate

$$\gamma^2(p^2\|\kappa v;K\|^2+\|\nabla_x(\kappa v);K\|^2) \\ \leq c(\|\kappa L(D_x,\tau)v;K\|^2+p^2\|\psi v;K\|^2+\|\nabla(\psi v);K\|^2+p^2\|\psi v;\partial K\|^2). \tag{3.12}$$

In (3.12) we replace v by the function $y\mapsto V_\varepsilon(x)=u(\varepsilon^{-1}x)$, where $N(x,D_x)u=0$ on ∂K, and instead of τ we take $\tau/(\varepsilon p)$ with $\varepsilon>0$. (Since the coefficients of the differential operator $N(x,D_x)$ only depend (linearly) on the unit normal $\nu(x)$ to ∂K, this substitution is possible.) Then (3.12) takes the form
(3.13)
$$\gamma^2\varepsilon^{-2}p^{-2}(\varepsilon^{-2}\|\kappa V_\varepsilon;K\|^2+\|\nabla_x(\kappa V_\varepsilon);K\|^2) \\ \leq c(\|\kappa L(D_x,\theta/\varepsilon)V_\varepsilon;K\|^2+\varepsilon^{-2}\|\psi V_\varepsilon;K\|^2+\|\nabla(\psi V_\varepsilon);K\|^2+\varepsilon^{-2}\|\psi V_\varepsilon;\partial K\|^2).$$

Making the change of variables $x \mapsto \eta = \varepsilon^{-1}x$ and multiplying by $\varepsilon^{4-(n-d)}$, we obtain

$$\begin{aligned}(3.14)\qquad &\gamma^2 p^{-2}(\|\kappa_\varepsilon u; K\|^2 + \|\nabla(\kappa_\varepsilon u); K\|^2)\\ &\quad \le c(\|\kappa_\varepsilon L(D_\eta,\theta)u; K\|^2 + \varepsilon^2\{\|\psi_\varepsilon u; K\|^2 + \|\nabla(\psi_\varepsilon u); K\|^2\} + \varepsilon\|\psi_\varepsilon u; \partial K\|^2),\end{aligned}$$

where $\kappa_\varepsilon(\eta) = \kappa(\varepsilon\eta)$, $\psi_\varepsilon(\eta) = \psi(\varepsilon\eta)$. We multiply (3.14) by $\varepsilon^{-2\beta}$, set $\varepsilon = 2^{-j}$, $j = 1, 2, \dots$, and sum all the inequalities. The sum of the terms $\varepsilon^{1-2\beta}\|\psi_\varepsilon u; \partial K\|^2$ with $\varepsilon = 2^{-j} < \gamma^2/2^4p^2c$ does not exceed $(\gamma^2/2^2p^2)\|\kappa_\infty u; H^1_\beta(K;p)\|^2$. Therefore, this sum can be moved to the left-hand side of the resulting inequality. The same is true for the terms coming from the expression $\varepsilon^2\{\|\psi_\varepsilon u; K\|^2 + \|\nabla(\psi_\varepsilon u); K\|^2\}$. Thus, we arrive at (3.3). □

If the commutator $[N(x,D_x),\kappa]$ vanishes, on the right-hand side of (3.3) the norm $\|\cdot; H^0_{\beta-1/2}(\partial K\|$ can be dropped. For instance, consider the case of $m = 1$, with the operators

$$(3.15)\qquad L(D_x, D_t) := \partial_t^2 - \Delta_x \quad\text{and}\quad N(x, D_x) := \partial_\nu.$$

Proposition 3.2. *Let L and N be as in* (3.15). *If $u \in C_c^\infty(\bar K \setminus O)$ is such that $N(x,D_x)u|\partial K = 0$, then*

$$\begin{aligned}(3.16)\qquad &(\gamma/p)^2\|\kappa_\infty u; H^1_\beta(K;1))\|^2\\ &\quad \le c\{\|\kappa_\infty L(D_x,\theta)u; H^0_\beta(K)\|^2 + \|\Psi u; H^1_{\beta-1}(K;1)\|^2\}\end{aligned}$$

with constant c independent of $\tau = \sigma - i\gamma$, where $\sigma \in \mathbb{R}$, $\gamma > 0$, and $p = |\tau|$.

Proof. Since $N(x,D_x)(\kappa u) = \partial_\nu u = 0$, Proposition 2.1 implies that

$$\gamma^2(p^2\|\kappa v; K\|^2 + \|\nabla_x(\kappa v); K\|^2) \le c(\|L(D_x,\tau)(\kappa v); K\|^2).$$

Now, we deduce (3.16) in the same way as (3.3) was deduced from (3.10). □

3.2. Estimates near the vertex of the cone. The boundary-value problem

$$(3.17)\qquad L(D_\eta,\theta)u = f,\ \ \eta \in K;\quad N(\eta, D_\eta)u = 0,\ \ \eta \in \partial K \setminus 0,$$

is elliptic. We shall use some results about elliptic boundary-value problems in domains with conical points (see, e.g., [7]).

We introduce an operator pencil $\mathfrak{A}$ in $\Omega = K \cap S^{n-1}$ by the rule

$$(3.18)\qquad \mathbb{C} \ni \lambda \mapsto \mathfrak{A}(\lambda) = \{|x|^{2-i\lambda}P(D_x)|x|^{i\lambda}, |x|^{1-i\lambda}N(x,D_x)|x|^{i\lambda}\}.$$

The map $\mathfrak{A}(\lambda) : H^2(\Omega) \to L_2(\Omega) \times H^{1/2}(\partial\Omega)$ is an isomorphism for all $\lambda \in \mathbb{C}$ except for the normal eigenvalues. Each strip $|\Im\lambda| <$ const contains finitely many points of the spectrum of $\mathfrak{A}(\lambda)$.

Proposition 3.3 (see [7, Theorem 4.1.2]). *For $\beta \in \mathbb{R}$, suppose that the line $\Im\lambda = \beta - 2 + n/2$ is free from the spectrum of $\mathfrak{A}$. Then for every $u \in H^2_\beta(K;1)$ satisfying $N(\eta, D_\eta)u = 0$ on ∂K we have the estimate*

$$(3.19)\qquad \|\chi u; H^2_\beta(K;1)\|^2 \le c\{\|\chi L(D_\eta,\theta)u; H^0_\beta(K)\|^2 + \|\psi u; H^1_\beta(K;1)\|^2\},$$

where $\chi, \psi \in C_c^\infty(\mathbb{R}^n)$, $\chi\psi = \chi$, and $\chi = 1$ near the vertex of K; the constant c is independent of $\theta = \delta/p$.

For $n = 2$ we shall need "nonhomogeneous" norms. Let $H^{2,0}_\beta(K)$ denote the space with the norm

$$\|u; H^{2,0}_\beta(K)\| = \|r^{\beta-1}u; L_2(K)\|^2 + \sum_{0<|\alpha|\le 2}\|r^{\beta-2+|\alpha|}D^\alpha u; L_2(K)\|^2.$$

Before proceeding to problem (3.17), we prove two lemmas. The first of them is contained in [7, Proposition 4.5.2].

Lemma 3.4. *Let $n=2$, and let $u \in H^{2,0}_\beta(K)$ with $\beta<1$. Then the limit*

$$u(0)=\lim_{r\to 0+}|\Omega|^{-1}\int_\Omega u(r,\omega)\,d\omega$$

(where $\Omega = K\cap S^1$ and $|\Omega|$ is the length of the arc Ω on S^1) exists. Moreover, $\chi u = \chi(v+u(0))$, where $v\in H^2_\beta(K)$, $\chi\in C_c^\infty(\mathbb{R}^2)$, and $\chi=1$ in a neighborhood of the origin. We have

$$\|r^{\beta-2}(u-u(0)); L_2(K\cap\{|y|\le 1\})\| \le c\|r^{\beta-1}\nabla u; L_2(K\cap\{|y|\le 1\})\|, \tag{3.20}$$

$$|u(0)|\le c\{\|r^{\beta-1}\nabla u; L_2(K\cap\{|y|\le 1\})\| + \|u; L_2(K\cap\{1/2\le|y|\le 1\})\|\}. \tag{3.21}$$

If $\beta\le 0$, then $u(0)=0$.

Lemma 3.5. *Under the assumptions of Lemma 3.4, let $\varepsilon\in(0,1)$. Then*

$$\begin{aligned}|u(0)|\le \varepsilon&\|r^{\beta-1}\nabla u; L_2(K\cap\{|y|\le 1\})\|\\ &+C_\varepsilon\|u; L_2(K\cap\{\varepsilon^{1/(1-\beta)}/2\le|y|\le 1\})\|.\end{aligned} \tag{3.22}$$

Proof. Applying (3.21) to the function $y\mapsto w(y)=u(\varepsilon y)$, we obtain

$$\begin{aligned}|u(0)|^2 &\le c\Big\{\varepsilon^2\int_{K\cap\{|x|\le 1\}}|x|^{2\beta-2}|\nabla u(\varepsilon y)|^2\,dx+\int_{K\cap\{1/2\le|x|\le 1\}}|u(\varepsilon x)|^2\,dx\Big\}\\ &\le c\Big\{\varepsilon^{2-2\beta)}\int_{K\cap\{|x|\le 1\}}|x|^{2\beta-2}|\nabla u(x)|^2\,dx+\varepsilon^{-2}\int_{K\cap\{\varepsilon/2\le|x|\le 1\}}|u(x)|^2\,dx\Big\}.\end{aligned}$$

It remains to replace ε by $\varepsilon^{1/(1-\beta)}$. □

Now we are ready to obtain "nonhomogeneous" estimates of solutions of problem (3.17) in a neighborhood of the vertex of K.

Proposition 3.6. *Suppose $n=2$, $u\in H^{2,0}_\beta(K)$ with $\beta<1$, and $N(\eta,D_\eta)u=0$ on ∂K. If the line $\Im\lambda=\beta-1$ contains no eigenvalues of the pencil $\mathfrak{A}$, then for some $\delta>0$ we have the estimate*

$$\|\chi u; H^{2,0}_\beta(K)\|\le c\{\|\chi L(D_\eta,\theta)u; H^0_\beta(K)\|+\|u; H^1(K\cap\{\delta\le|\eta|\le 2\})\|\} \tag{3.23}$$

with constant c independent of u and θ; here $\chi\in C_c^\infty(\mathbb{R}^2)$, $\chi(\eta)=1$ for $|\eta|\le 1$, and $\chi(\eta)=0$ for $|\eta|\ge 3/2$.

Proof. Let $\psi\in C_c^\infty(\mathbb{R}^2)$ be a cut-off function such that $\psi\chi=\chi$. By Lemma 3.4, $\psi u = \psi v+\psi u(0)$ with $\psi v\in H^2_\beta(K)$. Since the line $\Im\lambda=\beta-1$ is free from the spectrum of $\mathfrak{A}$, we have

$$\begin{aligned}&\|\chi v; H^2_\beta(K)\|\\ &\quad\le c\{\|\chi L(D_\eta,\theta)v; H^0_\beta(K)\|+\|\chi N(\eta,D_\eta)v; H^{1/2}_\beta(\partial K)\|+\|\psi v; H^1_\beta(K)\|\}.\end{aligned} \tag{3.24}$$

Observe that

$$\begin{aligned}\chi N(\eta,D_\eta)v&=\chi N(\eta,D_\eta)u-\chi N(\eta,D_\eta)u(0)=-\chi N(\eta,D_\eta)u(0),\\ \chi L(D_\eta,\theta)v&=\chi L(D_\eta,\theta)u+\chi L(D_\eta,\theta)u(0).\end{aligned}$$

Therefore, (3.24) implies the estimates

$$\begin{aligned}\|\chi u; H^{2,0}_\beta(K)\|&\le c(\|\chi v; H^2_\beta(K)\|+\|\chi u(0); H^{2,0}_\beta(K)\|)\\ &\le c\{\|\chi L(D_\eta,\theta)u; H^0_\beta(K)\|+\|\psi v; H^1_\beta(K)\|\\ &\qquad+\|\chi L(D_\eta,\theta)u(0); H^0_\beta(K)\|+\|\chi N(\eta,D_\eta)u(0); H^{1/2}_\beta(\partial K)\|\}\\ &\le c\{\|\chi L(D_\eta,\theta)u; H^0_\beta(K)\|+\|\psi v; H^1_\beta(K)\|+|u(0)|\}.\end{aligned} \tag{3.25}$$

Recall that if $\beta \le 0$, then $u(0)=0$. Using (3.20), we obtain

$$
\begin{aligned}
\|\psi v; H^1_\beta(K)\|^2 &\le c\int_{K\cap\{|x|\le 2\}} (|v|^2 r^{2\beta-2} + |\nabla u|^2 r^{2\beta})\,dx \\
&\le c\Big\{\varepsilon^2\int_{K\cap\{|x|\le\varepsilon\}} (|u-u(0)|^2 r^{2\beta-4} + |\nabla u|^2 r^{2\beta-2})\,dx \\
&\qquad + \int_{K\cap\{\varepsilon\le|x|\le 2\}} (|u-u(0)|^2 r^{2\beta-2} + |\nabla u|^2 r^{2\beta})\,dx\Big\} \\
&\le c\Big\{\varepsilon^2\int_{K\cap\{|x|\le 1\}} |\nabla u|^2 r^{2\beta-2}\,dx + |u(0)|^2\Big\} + C_\varepsilon\|u; H^{q+1}(K\cap\{\varepsilon\le|x|\le 2\})\|^2 \\
&\le c\{\varepsilon^2\|\chi u; H^{2,0}_\beta(K)\|^2 + |u(0)|^2\} + C_\varepsilon\|u; H^1(K\cap\{\varepsilon\le|x|\le 2\})\|^2.
\end{aligned}
$$

Taking a sufficiently small ε, we bound $|u(0)|$ with the help of inequality (3.22). Now we can rewrite (3.25) in the form

$$
\begin{aligned}
&\|\chi u; H^{2,0}_\beta(K)\| \\
&\quad\le c\|\chi L(D_\eta,\theta)u; H^0_\beta(K)\| + \varepsilon\|\chi u; H^{2,0}_\beta(K)\| + C_\varepsilon\|u; H^1(K\cap\{\delta\le|\eta\|\le 2\})\|.
\end{aligned}
$$

Transferring the term with factor ε to the left, we arrive at (3.23). $\square$

3.3. Global estimates.

Proposition 3.7. *Suppose the line $\Im\lambda = \beta - 2 + n/2$ contains no eigenvalues of the pencil $\mathfrak{A}$. Then, under the condition $\beta \le 1/2$, for any function $v \in H^1(K)\cap H^2_\beta(K)$ satisfying $N(x,D_x)v = 0$ on ∂K we have*

$$
\begin{aligned}
&\|\chi_p v; H^2_\beta(K;p)\|^2 + \gamma^2\|v; H^1_\beta(K;p)\|^2 \\
&\qquad\le c(\|L(D_x,\tau)v; H^0_\beta(K)\|^2 + (p^{2-2\beta}/\gamma^2)\|L(D_x,\tau)v; K\|^2)
\end{aligned}
\tag{3.26}
$$

with constant c independent of $\tau=\sigma-i\gamma$, where $\sigma\in\mathbb{R}$ and $\gamma>0$. If $1/2<\beta\le 1$, then inequality (3.26) remains valid with $p^{2\beta}/\gamma^{4\beta}$ in place of $p^{2-2\beta}/\gamma^2$ on the right-hand side.

Proof. We add estimates (3.3) and (3.19). We can supress the cut-off function κ_∞ on the left-hand side and the norm $\|\psi u; H^1_\beta(K;1)\|$ on the right-hand side of the resulting inequality. Therefore,

$$
\begin{aligned}
&\|\chi u; H^2_\beta(K;1)\|^2 + (\gamma/p)^2\|u; H^1_\beta(K;1)\|^2 \\
&\qquad\le c\{\|L(D_\eta,\theta)u; H^0_\beta\|^2 + \|\Psi u; H^1_{\beta-1}(K;1)\|^2 + \|\Psi u; H^0_{\beta-1/2}(\partial K)\|^2\}.
\end{aligned}
\tag{3.27}
$$

If $\beta\le 1/2$, then

$$
\begin{aligned}
\|\Psi u; H^0_{\beta-1/2}(\partial K)\|^2 &\le \int_{\{\eta\in\partial K: 0<c_1<|\eta|\}} |\eta|^{2(\beta-1/2)}|u(\eta)|^2\,d\eta \le c\|u;\partial K\|^2 \\
&\le c(\|u;K\|^2 + \|\nabla u; K\|^2) \le c(p/\gamma)^2\|L(D_\eta,\theta)u; K\|^2
\end{aligned}
\tag{3.28}
$$

(this follows from the well-known inclusion $H^1(K)\subset L_2(\partial K)$ and estimate (2.6) with $\theta = (\sigma - i\gamma)/p$ in place of $\sigma - i\gamma$ and γ/p in place of γ). Moreover,

$$
\begin{aligned}
\|\Psi u; H^1_{\beta-1}(K;1)\|^2 &\le c\int_{0<c_1<|\eta|} |\eta|^{2(\beta-1)}(|\nabla u|^2 + |\eta|^{-2}|u|^2 + |u|^2)\,d\eta \\
&\le c\int_K (|\nabla u|^2 + |u|^2)\,d\eta \le c(p/\gamma)^2\|L(D_\eta,\theta)u; K\|^2.
\end{aligned}
\tag{3.29}
$$

Inequalities (3.27), (3.28), and (3.29) imply

$$\|\chi u; H^2_\beta(K;1)\|^2 + (\gamma/p)^2\|u; H^1_\beta(K;1)\|^2 \le c\{\|L(D_\eta,\theta)u; H^0_\beta(K)\|^2 + (p/\gamma)^2\|L(D_\eta,\theta)u; L_2(K)\|^2\}. \tag{3.30}$$

Here we put $x = p^{-1}\eta$, $\chi_p(x) = \chi(px)$, and $v(x) = u(px)$. Observe that

$$N(\eta, D_\eta)u|\partial K = 0 \iff N(x, D_x)v|\partial K = 0.$$

As a result, we obtain (3.26).

Now, suppose that $1/2 < \beta \le 1$. Formula (3.29) is still true, whereas (3.28) must be modified. We have

$$\begin{aligned}\|\Psi u; H^0_{\beta-1/2}(\partial K)\|^2 &\le \int_{\{\eta\in\partial K:|\eta|<C_1p^2/\gamma^2\}} |\eta|^{2(\beta-1/2)}|u(\eta)|^2\,d\eta \\ &\le C_2(p^2/\gamma^2)^{2\beta-1}\|u;\partial K\|^2 \le C_3(p^2/\gamma^2)^{2\beta-1}(\|u;K\|^2 + \|\nabla u;K\|^2) \\ &\le C_4(p^2/\gamma^2)^{2\beta}\|L(D_\eta,\theta)u;K\|^2.\end{aligned} \tag{3.31}$$

Thus, we arrive at inequality (3.30) with $(p/\gamma)^2 + (p/\gamma)^{4\beta}$ in place of $(p/\gamma)^2$ on the right. Obviously, $(p/\gamma)^2 + (p/\gamma)^{4\beta} < 2(p/\gamma)^{4\beta}$ for $\beta > 1/2$. As before, passage to the variables x, $\chi_p(x)$, and $v(x)$ completes the proof. □

Turning to nonhomogeneous norms, we introduce the space $H^{2,0}_\beta(K;p)$ with the norm

$$\|v; H^{2,0}_\beta(K;p)\| = \left(\int \sum_{|\alpha|=1}^{2} |x|^{2\beta-4+2|\alpha|}|D^\alpha_x v|^2\,dx + p^2\|v; H^1_\beta(K;p)\|^2\right)^{1/2}. \tag{3.32}$$

We have

$$\|v; H^2_\beta(K;p)\|^2 = \|v; H^{2,0}_\beta(K;p)\|^2 + \int |x|^{2\beta-4}|v(x)^2\,dx.$$

Proposition 3.8. *Suppose that $n = 2$, $\beta \le 1$, and the line $\Im\lambda = \beta - 1$ is free from the spectrum of the pencil $\mathfrak{A}$. Then, under the condition $\beta \le 1/2$, for any function $v \in C^\infty_c(\bar K)$ satisfying $N(x, D_x)v = 0$ on ∂K we have the estimate*

$$\|\chi_p v; H^{2,0}_\beta(K;p)\|^2 + \gamma^2\|v; H^1_\beta(K;p)\|^2 \le c(\|L(D_x,\tau)v; H^0_\beta(K)\|^2 + (p^{2-2\beta}/\gamma^2)\|L(D_x,\tau)v;K\|^2) \tag{3.33}$$

with constant c independent of τ. If $1/2 < \beta \le 1$, inequality (3.33) *remains valid with $p^{2\beta}/\gamma^{4\beta}$ in place of $p^{2-2\beta}/\gamma^2$ on the right.*

Proof. Assume for instance that $\beta \le 1/2$. Using (3.23) rather than (3.19) and arguing as in the proof of Proposition 3.7, we obtain

$$\|\chi u; H^{2,0}_\beta(K;1)\|^2 + (\gamma/p)^2\|u; H^1_\beta(K;1)\|^2 \le c\{\|L(D_\eta,\theta)u; H^0_\beta(K)\|^2 + (p/\gamma)^2\|L(D_\eta,\theta)u; L_2(K)\|^2\}. \tag{3.34}$$

After the change of variables $\eta \mapsto x = p^{-1}\eta$, this inequality takes the form

$$\begin{aligned}&\int \sum_{|\alpha|=1}^{2} |x|^{2\beta-4+2|\alpha|}|D^\alpha_x(\chi_p v)|^2\,dx + p^2\int |x|^{2\beta-2}|\chi_p v|^2\,dx + \gamma^2\|v; H^1_\beta(K;p)\|^2 \\ &\qquad \le c(\|L(D_x,\tau)v; H^0_\beta(K)\|^2 + (p^{2-2\beta}/\gamma^2)\|L(D_x,\tau)v;K\|^2).\end{aligned} \tag{3.35}$$

Since $p|x| \le \text{const}$ for $x \in \operatorname{supp}\chi_p$, we can add the remaining terms to the left-hand side of (3.35) to get $\|\chi_p v; H^{2,0}_\beta(K;p)\|^2$. □

§4. The problem with parameter in a cone: A scale of weighted spaces

4.1. The operator of the problem in weighted spaces. We define scales of function spaces suggested by Propositions 3.7 and 3.8. In the case where $n \geq 3$, the space $DH_\beta(K;p)$ is endowed with the norm

$$\|v; DH_\beta(K;p)\| = (\|\chi_p v; H^2_\beta(K;p)\|^2 + \gamma^2 \|v; H^1_\beta(K;p)\|^2)^{1/2}. \tag{4.1}$$

If $n = 2$, then, by definition,

$$\|v; DH_\beta(K;p)\| = (\|\chi_p v; H^{2,0}_\beta(K;p)\|^2 + \gamma^2 \|v; H^1_\beta(K;p)\|^2)^{1/2}. \tag{4.2}$$

For $\beta \leq 1/2$, we denote by $RH_\beta(K;p)$ the space with the norm

$$\|f; RH_\beta(K;p)\| = (\|f; H^0_\beta(K)\|^2 + (p^{2-2\beta}/\gamma^2)\|f; L_2(K)\|^2)^{1/2}. \tag{4.3}$$

If $1/2 < \beta \leq 1$, we put

$$\|f; RH_\beta(K;p)\| = (\|f; H^0_\beta(K)\|^2 + (p^{2\beta}/\gamma^{4\beta})\|f; L_2(K)\|^2)^{1/2}. \tag{4.4}$$

We introduce the (unbounded) operator

$$v \mapsto L(D_x, \tau)v \tag{4.5}$$

in $RH_\beta(K;p)$, with domain

$$\big\{v \in C^\infty(\bar{K} \setminus O) \cap DH_\beta(K;p) \cap H^1(K) : \\ L(D_x,\tau)v \in RH_\beta(K;p), N(x, D_x)v = 0 \text{ on } \partial K\big\}.$$

The operator (4.5) admits closure. We denote by $A_\beta(\tau)$ this closure and by $D(A_\beta(\tau))$ its domain. The next statement follows from Propositions 3.7 and 3.8.

Proposition 4.1. *Suppose $\beta \leq 1$ and the line $\Im\lambda = \beta - 2 + n/2$ contains no eigenvalues of the pencil $\mathfrak{A}$. Then*
1) $D(A_\beta(\tau)) \subset D(A(\tau))$;
2) $\ker A_\beta(\tau) = 0$;
3) $\operatorname{im} A_\beta(\tau)$ *is closed in* $RH_\beta(K;p)$.

In order to describe $\operatorname{im} A_\beta(\tau)$, we need information about solutions of the homogeneous problem (2.5) (with $f \equiv 0$). We present this information in the next subsection, mainly restricting ourselves to formulations. For the proofs, we refer the reader, e.g., to [7].

4.2. On solutions of the homogeneous problem with parameter in the cone. Let $\mathbb{C} \ni \lambda \mapsto \mathfrak{A}(\lambda)$ be the pencil defined in (3.18). Consider the function

$$u(y) = r^{i\lambda} \sum_{q=0}^{k} \frac{1}{q!}(i \log r)^q \varphi^{(k-q)}(\omega), \tag{4.6}$$

where $r = |x|$, $\omega = x/|x|$. This function is a solution of the boundary-value problem

$$\begin{cases} P(D_x, 0)u(x) = 0 & \text{if } x \in K, \\ N(x, D_x)u(x) = 0 & \text{if } x \in \partial K \end{cases} \tag{4.7}$$

if and only if λ is an eigenvalue of $\mathfrak{A}$ and $\varphi^{(0)}, \dots, \varphi^{(k-q)}$ is a Jordan chain corresponding to λ ($\varphi^{(0)}$ is an eigenvector and $\varphi^{(1)}, \dots, \varphi^{(k)}$ are generalized eigenvectors). Any solution of the form (4.6) is called a *power solution*. Let $\kappa_1 \geq \cdots \geq \kappa_J$ be the partial multiplicities of an eigenvalue λ_0, and let $\{\varphi^{(0,j)}, \dots, \varphi^{(\kappa_j - 1, j)}; j = 1, \dots, J\}$ be a canonical system of Jordan chains. The functions

$$u^{(k,j)}(y) = r^{i\lambda_0} \sum_{q=0}^{k} \frac{1}{q!}(i \log r)^q \varphi^{(k-q,j)}(\omega), \tag{4.8}$$

where $k = 0, \dots, \kappa_j - 1$, $j = 1, \dots, J$, constitute a basis in the space of power solutions corresponding to λ_0.

The operator pencil (3.18) can be written as $\mathfrak{A}(\lambda) = \{\mathrm{P}(\lambda), \mathrm{N}(\lambda)\}$. For φ and ψ in $H^2(\Omega)$, we have the Green formula

$$(4.9)\qquad \begin{aligned}&(\mathrm{P}(\lambda)\phi, \psi)_\Omega + (\mathrm{N}(\lambda)\phi, \psi)_{\partial\Omega}\\ &\quad= (\phi, \mathrm{P}(\bar\lambda + i(n-2))\psi)_\Omega + (\phi, \mathrm{N}(\bar\lambda + i(n-2))\psi)_{\partial\Omega},\end{aligned}$$

which follows from the Green formula (2.2). Therefore, $\mathfrak{A}(\lambda)^* = \mathfrak{A}(\bar\lambda + i(n-2))$, where $\mathfrak{A}(\lambda)^*$ stands for the adjoint operator to $\mathfrak{A}(\lambda)$ relative to the Green formula (4.9). The spectrum of the pencil $\mathfrak{A}(\lambda)$ is symmetric with respect to the line $\Im\lambda = (n-2)/2$.

We introduce the pencil

$$\mathbb{C} \ni \lambda \mapsto \mathfrak{A}^*(\lambda) := [\mathfrak{A}(\bar\lambda)]^* = \mathfrak{A}(\lambda + i(n-2)).$$

If λ_0 is an eigenvalue of $\mathfrak{A}$, then $\bar\lambda_0$ is an eigenvalue of $\mathfrak{A}^*$, and the geometric and algebraic multiplicities of λ_0 and $\bar\lambda_0$ coincide. Canonical systems of Jordan chains $\{\varphi^{(0,j)}, \dots, \varphi^{(\kappa_j-1,j)}; j = 1, \dots, J\}$ and $\{\psi^{(0,j)}, \dots, \psi^{(\kappa_j-1,j)}; j = 1, \dots, J\}$ corresponding to λ_0 and $\bar\lambda_0$ can be chosen to satisfy the orthogonality and normalization condition

$$(4.10)\qquad \begin{aligned}&\sum_{p=0}^{\nu}\sum_{q=0}^{k} \frac{1}{(\nu+k+1-p-q)!}\\ &\qquad\times \bigl\{(\partial_\lambda^{\nu+k+1-p-q}\mathrm{P}(\lambda_0)\varphi^{(q,\sigma)}, \psi^{(p,\zeta)})_\Omega + (\partial_\lambda^{\nu+k+1-p-q}\mathrm{N}(\lambda_0)\varphi^{(q,\sigma)}, \psi^{(p,\zeta)})_{\partial\Omega}\bigr\}\\ &\quad= \delta_{\sigma,\zeta}\delta_{\kappa_\sigma-k-1,\nu},\end{aligned}$$

where $\delta_{p,q}$ is the Kronecker symbol, $\sigma, \zeta = 1, \dots, J$, $\nu = 0, \dots, \kappa_\zeta - 1$, and $k = 0, \dots, \kappa_\sigma - 1$. The functions

$$(4.11)\qquad v^{(k,j)}(y) = r^{i\bar\lambda_0-(n-2)}\sum_{q=0}^{k}(q!)^{-1}(i\log r)^q\psi^{(k-q,j)}(\omega)$$

form a basis in the space of power solutions of problem (4.7) corresponding to $\bar\lambda_0 + i(n-2)$. If condition (4.10) is fulfilled, then the bases (4.8) and (4.11) are said to *match*.

Consider the homogeneous problem

$$(4.12)\qquad L(D_x, \tau)u = 0 \text{ in } K; \quad N(x, D_x)u = 0 \text{ on } \partial K.$$

To be specific, first we assume that the line $\Im\lambda = (n-2)/2$ is free from the spectrum of $\mathfrak{A}$. (In general, this assumption may fail if $n = 2$. The modifications needed for handling such problems will be indicated later.) Let λ_μ be an eigenvalue of the pencil $\mathfrak{A}$ such that $\Im\lambda_\mu < (n-2)/2$. We denote by $\{u_\mu^{(k,j)}\}$ and $\{v_\mu^{k,j}\}$ some matching bases of power solutions of problem (4.7) corresponding to λ_μ and $\bar\lambda_\mu + i(n-2)$. Substituting the functions $u_\mu^{(k,j)}$ in (4.12) and compensating the discrepancies successively (see [7]), we construct the formal series

$$(4.13)\qquad U_\mu^{(k,j)}(x, \tau) = \sum_{q=0}^{\infty} r^{i\lambda_\mu+q}P_q^{(k,j)}(\omega, \tau, \log r)$$

satisfying (4.12). Here the $P_q^{(k,j)}$ are polynomials in $\log r$ and τ with coefficients smoothly depending on $\omega \in \bar\Omega$. Replacing $u_\mu^{(k,j)}$ by $v_\mu^{(k,j)}$, we obtain the formal series

$$(4.14)\qquad V_\mu^{(k,j)}(x, \tau) = \sum_{q=0}^{\infty} r^{i(\bar\lambda_\mu+i(n-2))+q}Q_q^{(k,j)}(\omega, \tau, \log r)$$

satisfying (4.12), where the properties of $Q_q^{(k,j)}$ are similar to those of $P_q^{(k,j)}$.

Proposition 4.2. *Let λ_μ with $\Im\lambda_\mu < (n-2)/2$ be an eigenvalue of $\mathfrak{A}$, and let $\{v_\mu^{(k,j)}\}$ be a basis in the space of power solutions corresponding to the eigenvalue $\bar{\lambda}_\mu + i(n-2)$ (see (4.11)). Then there exist functions $x \mapsto w_\mu^{(k,j)}(x,\tau)$ in $C^\infty(\bar{K} \setminus O)$ satisfying (4.12) and such that*

$$w_\mu^{(k,j)}(x,\tau) = \chi V_{\mu,T}^{(k,j)}(x,\tau) + \rho(x,\tau), \tag{4.15}$$

where $V_{\mu,T}^{(k,j)}(x,\tau)$ is the Tth partial sum of the series (4.14) with sufficiently large T. The function ρ depends on k, j, μ, T, and χ, belongs to $C^\infty(\bar{K} \setminus O)$, and satisfies $\rho(x,\tau) = O(|x|^h)$ with $h = \min\{-\Im\lambda_\mu : \Im\lambda_\mu < (n-2)/2\}$ as $x \to 0$ and τ is fixed. Moreover, $(1-\chi)\rho \in H_\beta^2(K;p)$ for any $\beta \in \mathbb{R}$. The functions $w_\mu^{(k,j)}(\cdot,\tau)$ are unique and do not depend on the choice of T and χ.

Proof. We outline the proof, which proceeds in several steps.

A) We check the uniqueness of $w_\mu^{(k,j)}$. Suppose there exists another function $\tilde{w}_\mu^{(k,j)}$ with representation similar to (4.15), $\tilde{w}_\mu^{(k,j)} = \tilde{\chi} V_{\mu,\tilde{T}}^{(k,j)} + \tilde{\rho}$. Then the difference $w_\mu^{(k,j)} - \tilde{w}_\mu^{(k,j)}$ is in $\ker A(\tau) = 0$, whence $w_\mu^{(k,j)} = \tilde{w}_\mu^{(k,j)}$.

B) We choose a function for the role of ρ; later we shall verify its properties. Introduce the functions

$$f_T := L(D_x,\tau)\chi V_{\mu,T}^{(k,j)} = \chi L V_{\mu,T}^{(k,j)} + [L,\chi] V_{\mu,T}^{(k,j)},$$
$$g_T := N(x,D_x)\chi V_{\mu,T}^{(k,j)} = \chi N V_{\mu,T}^{(k,j)} + [N,\chi] V_{\mu,T}^{(k,j)}.$$

It is clear that $\operatorname{supp} f_T$ and $\operatorname{supp} g_T$ are compact and that for sufficiently large T the functions f_T and g_T decay rapidly as $y \to 0$. We look for a function ρ satisfying

$$L(D_x,\tau)\rho = -f_T,\ x \in K; \quad N(x,D_x)\rho = -g_T,\ x \in \partial K. \tag{4.16}$$

To prove the existence of ρ, we are going to use Theorem 2.5; therefore, we must pass from (4.16) to a problem with a homogeneous boundary condition. For this, we consider the elliptic boundary-value problem

$$P(D_x)v = F,\ x \in K; \quad N(x,D_x)v = -g_T,\ x \in \partial K, \tag{4.17}$$

where $F \in C_c^\infty(K)$.

In what follows we use some known results on elliptic problems in a cone (see, e.g., [7]). Problem (4.17) has a unique solution v in $H_1^2(K)$. In order that $(1-\chi)v$ belong to $H_\beta^2(K)$ for a given β, the function F must be chosen in such a way that the right-hand side of problem (4.17) satisfy a finite number of orthogonality conditions; these conditions can be chosen to nullify the part of the asymptotics of v at infinity that does not belong to $H_\beta^2(K)$. (As a matter of fact, the coefficients in the asymptotics are inner products that involve the right-hand side of problem (4.17) and some special solutions of the corresponding homogeneous problem.) Given β', we obtain $\chi v \in H_{\beta'}^2(K)$ under a few additional orthogonality conditions imposed on the right-hand side of (4.17) (here we assume T to be sufficiently large to ensure that $\chi g_T \in H_{\beta'}^{1/2}(\partial K)$). For the function $\tilde{\rho} := \rho + v$, we have the boundary-value problem

$$L(D_x,\tau)\tilde{\rho} = -f_T + F + \tau^2 v,\ \ x \in K; \quad N(x,D_x)\tilde{\rho} = 0,\ \ x \in \partial K. \tag{4.18}$$

By Theorem 2.5, problem (4.18) admits a unique solution $\tilde{\rho} \in D(A(\tau))$.

C) We show that the function $w_\mu^{(k,j)}$ defined by (4.15) with $\rho := \tilde{\rho} - v$ possesses the properties claimed in the proposition.

Problem (4.18) with fixed τ is elliptic, and its right-hand side is in $C^\infty(\bar{K} \setminus O)$. Therefore, $\tilde{\rho} \in C^\infty(\bar{K} \setminus O)$. Moreover, since $-f_T + F + \tau^2 v$ decays rapidly as $|x| \to 0$, we have $\tilde{\rho}(x) = O(|x|^h)$. Clearly, the same is true for ρ. Finally, to prove the relation $(1-\chi)w_\mu^{(k,j)} \in H^2_\beta(K;p)$ for any $\beta \in \mathbb{R}$, we can employ the same argument as in the case of the Dirichlet boundary condition (see [6], Proposition 4.3). $\square$

Remark 4.3. Since the operator $\{L(D_x, D_t), N(x, D_x)\}$ is invariant under the transformation $t \mapsto -t$, Proposition 4.2 remains true if $\tau = \sigma - i\gamma$ is replaced by $\bar{\tau} = \sigma + i\gamma$.

4.3. Description of $\operatorname{im} A_\beta(\tau)$**.** First, we recall some preliminary facts about the behavior of a strong solution of problem (2.13) near the vertex of K and near infinity. This is not a definitive result on asymptotics yet, because this information contains no remainder estimate uniform with respect to τ. For the time being, we keep the assumption that the line $\Im\lambda = (n-2)/2$ is free from the spectrum of $\mathfrak{A}$. Moreover, we assume that the same is true for the line $\Im\lambda = \beta - 2 + n/2$ with some $\beta < 1$. We denote by S_β the set of all eigenvalues of $\mathfrak{A}$ in the strip $\beta - 2 + n/2 < \Im\lambda < (n-2)/2$.

Lemma 4.4. *Suppose τ is fixed and $f \in RH_\beta(K;p)$. Then the solution of the equation $A(\tau)u = f$ admits the representation*

$$u = \sum_{S_\beta} c_\mu^{(k,j)} \chi U_{\mu,T}^{(k,j)} + \tilde{u}, \tag{4.19}$$

where $\chi\tilde{u} \in H^2_\beta(K;p)$ and $U_{\mu,T}^{(k,j)}$ is the Tth partial sum of the series (4.13) with sufficiently large T. The coefficients $c_\mu^{(k,j)}$ are continuous functionals on $RH_\beta(K;p)$ defined by

$$c_\mu^{(k,j)} = i(f, w_\mu^{(k,j)}(\cdot, \bar{\tau})), \tag{4.20}$$

where $w_\mu^{(k,j)}$ is given by (4.15), and $(\cdot,\cdot)$ stands for the (extended) inner product on $L_2(K)$. Finally, the bases of power solutions $u_\mu^{(k,j)}$ and $v_\mu^{(k,j)}$ used in (4.13) and (4.14) are matched in the sense of (4.10).

Moreover, if $(1-\chi)f \in H^0_{\beta'}(K)$ with $\beta' \geq \beta$, then $(1-\chi)u \in H^2_{\beta'}(K)$.

Proof. Obviously, $RH_\beta(K;p) \subset L_2(K)$. By Theorem 2.5, there exists a unique solution of the equation $A(\tau)u = f$. The representation (4.19) was obtained in the theory of elliptic boundary-value problems (see, e.g., [7]). To prove (4.20), we can invoke Proposition 4.2 and use the same argument as in [10] (see also [7]). The relation $(1-\chi)u \in H^2_{\beta'}(K)$ can be verified in the same way as in [6, Proposition 4.3]. $\square$

Proposition 4.5. *Suppose that the assumptions listed at the beginning of this subsection are fulfilled. Then*

$$\operatorname{im} A_\beta(\tau) = \{f \in RH_\beta(K;p) : (f, w_\mu^{(k,j)}(\cdot,\bar{\tau})) = 0 \textit{ for all } \lambda_\mu \in S_\beta\}.$$

Proof. If $f \in RH_\beta(K;p)$, then (4.19) is valid with $c_\mu^{(k,j)}$ defined by (4.20). Note that $\chi U_{\mu,T}^{(k,j)} \notin DH_\beta(K;p)$ for $\lambda_\mu \in S_\beta$. Therefore, $u \notin D(A_\beta(\tau))$ unless $c_\mu^{(k,j)} = 0$ for all $\lambda_\mu \in S_\beta$. This means that

$$\operatorname{im} A_\beta(\tau) \subset \{f \in RH_\beta(K;p) : (f, w_\mu^{(k,j)}(\cdot,\bar{\tau})) = 0 \text{ for all } \lambda_\mu \in S_\beta\}.$$

We prove the reverse inclusion. Given $f \in RH_\beta(K;p)$, take a sequence $\{f_n\} \subset C_c^\infty(K)$ such that $f_n \to f$ in $RH_\beta(K;p)$. By Lemma 4.4, for u_n satisfying $A(\tau)u_n = f_n$ we have the representation

$$u_n = i\chi \sum_{S_\beta} (f_n, w_\mu^{(k,j)}(\cdot,\bar{\tau}))\chi U_{\mu,T}^{(k,j)} + \tilde{u}_n$$

with $\chi\tilde{u}_n \in H^2_\beta(K;p)$. Choosing T sufficiently large, we can assume that the function $\partial K \ni x \to N(x, D_x)(\chi U_{\mu,T}^{(k,j)})$ decays rapidly as $y \to 0$. In the proof of Proposition 4.2

we obtained a function v that satisfies $N(x, D_x)v = -g_t$ on ∂K and decays rapidly near the vertex of K and near infinity. Now we put $g_T = -N(x, D_x)(\chi U^{(k,j)}_{\mu,T})$, denote the corresponding v by $\tilde U^{(k,j)}_{\mu,T}$, and introduce

$$v_n = u_n - i \sum_{S_\beta} (f_n, w^{(k,j)}_\mu(\cdot, \bar\tau))(\chi U^{(k,j)}_{\mu,T} - \tilde U^{(k,j)}_{\mu,T}).$$

Clearly, $N(x, D_x)v_n = 0$ on ∂K and $v_n \in D(A_\beta(\tau))$. Moreover,

$$L(D_x, \tau)v_n = f_n - i \sum_{S_\beta} (f_n, w^{(k,j)}_\mu(\cdot, \bar\tau))L(D_x, \tau)(\chi U^{(k,j)}_{\mu,T} - \tilde U^{(k,j)}_{\mu,T}). \tag{4.21}$$

The functionals $h \mapsto (h, w^{(k,j)}_\mu(\cdot, \bar\tau))$ are continuous on $RH_\beta(K; p)$. Therefore, as $n \to \infty$, the right-hand side of (4.21) tends to

$$f - i \sum_{S_\beta} (f, w^{(k,j)}_\mu(\cdot, \zeta, \bar\tau))L(D_y, \zeta, \tau)(\chi U^{(k,j)}_{\mu,T} - \tilde U^{(k,j)}_{\mu,T})$$

in $RH_\beta(K; p))$. By Proposition 3.7, we have

$$\|v_n; DH_\beta(K; p)\| \le c\|f_n; RH_\beta(K; p)\|,$$

so that the limit $v = \lim v_n$ exists in $DH_\beta(K; p)$, and hence, in $RH_\beta(K; p)$. It follows that $v \in D(A_\beta(\tau))$ and

$$A_\beta(\tau)v = f - i \sum_{S_\beta} (f, w^{(k,j)}_\mu(\cdot, \bar\tau))L(D_x, \tau)(\chi U^{(k,j)}_{\mu,T} - \tilde U^{(k,j)}_{\mu,T}).$$

If $(f, w^{(k,j)}_\mu(\cdot, \bar\tau))$=0 for all $\lambda_\mu \in S_\beta$, then $f \in \operatorname{im} A_\beta(\tau)$. □

The modifications needed in the case where $n = 2$ stem from the fact that the line $\Im\lambda = 0$ is not free from the spectrum of $\mathfrak{A}$. It is known (see, e.g., [11, 7]) that this line contains one eigenvalue $\lambda_0 = 0$. Its multiplicity is equal to $2m$. A canonical system of Jordan chains corresponding to λ_0 is of the form $\{\varphi^{(0,j)}_0, \varphi^{(1,j)}_0\}^m_{j=1}$, where $\varphi^{(0,j)}_0$ is an eigenvector and $\varphi^{(1,j)}_0$ is an associated eigenvector. There exist solutions $w^{(1,j)}_0(\cdot, \tau)$ of the homogeneous problem (4.12) that admit representations of the form (4.15) with $V^{(1,j)}_{0,T}$ on the right in place of $V^{(k,j)}_{\mu,T}$; the principal term of $V^{(1,j)}_{0,T}$ is $\varphi^{(1,j)}_0 + \varphi^{(0,j)}_0 \log |y|$. Let $U^{(0,j)}_0$ denote the formal series satisfying (4.12) with the principal term $\varphi^{(0,j)}_0$, $j = 1, \dots, l$. Lemma 4.4 is still valid, but the terms $c^{(0,j)}_0 U^{(0,j)}_{0,T}$ with $c^{(0,j)}_0 = i(f, w^{(1,j)}_0(\cdot, \bar\tau))$ for $j = 1, \dots, l$ must be added to the sum $\sum_{S_\beta}$ in (4.19). This enables us to describe $\operatorname{im} A_\beta(\tau)$ for $n = 2$ as well. Summarizing the results, we arrive at the following two theorems.

Let $\{\beta_k\}$ be the sequence of all numbers $1 > \beta_1 > \beta_2 > \cdots$ such that the line $\Im\lambda = \beta_k - 2 + n/2$ contains an eigenvalue of the pencil $\mathfrak{A}$. It is known that, for $n \ge 3$ and $\beta = 1$, the line $\Im\lambda = \beta - 2 + n/2 = (n - 2)/2$ is free from the spectrum of $\mathfrak{A}$ (see [7, 11]).

Theorem 4.6. *Assume that $n \ge 3$. If $\beta_1 < \beta \le 1$, then the equation*

$$A_\beta(\tau)u = f \in RH_\beta(K; p) \tag{4.22}$$

has a unique solution u for any f. The estimate

$$\|u; DH_\beta(K; p)\| \le c\|f; RH_\beta(K; p)\| \tag{4.23}$$

is valid with constant c independent of $\tau = \sigma - \gamma$, where $\sigma \in \mathbb{R}$ and $\gamma > 0$.

If $\beta_{q+1} < \beta < \beta_q$, *equation* (4.22) *is solvable if and only if*

$$(f, w_\mu^{(k,j)}(\cdot, \zeta, \bar{\tau})) = 0 \quad \textit{for all } \lambda_\mu \in S_\beta; \tag{4.24}$$

here, as before, S_β *denotes the set of all eigenvalues of the pencil* $\mathfrak{A}$ *in the strip* $\beta - 2 + n/2 < \Im\lambda < 1$. *The solution is unique and satisfies* (4.23).

Recall that the norms $\|\cdot; RH_\beta(K;p)\|$ for $\beta \le 1/2$ and for $1/2 < \beta \le 1$ are given by (4.3) and (4.4), respectively. Moreover, the definitions of $DH_\beta(K;p)$ are different for $n \ge 3$ and for $n = 2$ (see (4.1) and (4.2)).

Theorem 4.7. *Let* $n = 2$. *If* $\beta_1 < \beta < 1$, *then equation* (4.22) *has a solution for any* f. *If* $\beta_{q+1} < \beta < \beta_q$ *and* $\beta > -1/2$, *then equation* (4.22) *is solvable only under conditions* (4.24), *and if* $\beta \le -1/2$, *the requirements* $(f, w_0^{(1,j)}(\cdot, \zeta, \bar{\tau})) = 0$ *for* $j = 1, \dots, l$ *must be added to conditions* (4.24). *The solution is unique and satisfies* (4.23).

Since $D(A_\beta(\tau)) \subset D(A(\tau))$ by Proposition 4.1, we see that the solutions provided by Theorems 4.6 and 4.7 are strong solutions of problem (2.13). Thus, the above theorems describe conditions under which the strong solution belongs to $D(A_\beta(\tau))$.

§5. The asymptotics of solutions

5.1. The asymptotics of solutions of the problem in a cone. The change of variables

$$\eta = px, \quad U(\eta,\tau) = \hat{u}(p^{-1}\eta, \tau), \quad F(\eta,\tau) = p^{-2}\hat{f}(p^{-1}\eta, \tau)$$

transforms (2.5) to

$$\begin{cases} L(D_\eta, \theta)U(\eta,\tau) = F(\eta,\tau) & \text{if } \eta \in K, \\ N(\eta, D_\eta)\hat{u}(x,\tau) = 0 & \text{if } \eta \in \partial K, \end{cases} \tag{5.1}$$

where $\theta = \tau/p$; we also put $\bar{\theta} = \bar{\tau}/p$.

Let $F \in RH_\beta(K;1)$ with $\beta_{q+1} < \beta < \beta_q$. Then, as was shown in the proof of Proposition 4.5, the strong solution of (5.1) admits the representation

$$U = V + i\sum_{S_\beta}(F, w_\mu^{(k,j)}(\cdot, \bar{\theta}))(\chi U_{\mu,T}^{(k,j)}(\cdot,\theta) - \tilde{U}_{\mu,T}^{(k,j)}(\cdot,\theta)), \tag{5.2}$$

where V is the solution of the equation $A_\beta(\theta)V = F'$ with

$$F' := F - i\sum_{S_\beta}(F, w_\mu^{(k,j)}(\cdot, \bar{\theta}))L(D_\eta,\theta)(\chi U_{\mu,T}^{(k,j)}(\cdot,\theta) - \tilde{U}_{\mu,T}^{(k,j)}(\cdot,\theta)); \tag{5.3}$$

here by $\sum_{S_\beta}$ we mean the sum of all terms corresponding to the eigenvalues of $\mathfrak{A}$ in the strip $\beta - 2 + n/2 < \Im\lambda \le (n-2)/2$. Since

$$|(F, w_\mu^{(k,j)}(\cdot, \bar{\theta}))| \le c\|F; RH_\beta(K;1)\|, \tag{5.4}$$

we have

$$\|F'; RH_\beta(K;1)\| \le c\|F; RH_\beta(K;1)\| \tag{5.5}$$

with a constant c independent of θ. Now, Theorems 4.6 and 4.7 yield the estimate

$$\|V; DH_\beta(K,1)\| \le c\|F; RH_\beta(K;1)\|. \tag{5.6}$$

Obviously,

$$\left\| \sum_{S_\beta}(F, w_\mu^{(k,j)}(\cdot,\bar{\theta}))\tilde{U}_{\mu,T}^{(k,j)}(\cdot,\theta); DH_\beta(K;1)\right\| \le c\|F; RH_\beta(K;1)\|. \tag{5.7}$$

Thus, we arrive at the following assertion.

Theorem 5.1. *If $\beta_{k+1} < \beta < \beta_q$ and $F \in RH_\beta(K;1)$, then the strong solution U of problem* (5.1) *admits the representation*

$$U = i\chi \sum_{S_\beta} d_\mu^{(k,j)} U_{\mu,T}^{(k,j)}(\cdot,\theta) + \rho(\cdot,\theta), \tag{5.8}$$

where the coefficients

$$d_\mu^{(k,j)} = (F, w_\mu^{(k,j)}(\cdot,\bar{\theta})) \tag{5.9}$$

are continuous functionals on $RH_\beta(K;1)$, and the remainder $\rho(\cdot,\theta)$ satisfies

$$\|\rho(\cdot,\theta); DH_\beta(K;1)\| \le c\|F; RH_\beta(K;1)\| \tag{5.10}$$

with a constant c independent of θ.

5.2. On the problem in the cylinder $Q = K \times \mathbb{R}$. Here we formulate the results concerning the problem in Q and obtained from those on the problem in the cone K with the help of the inverse Fourier transformation. We restrict ourselves to the case where $n \ge 3$.

The space $H_\beta^s(Q)$ is the completion of the set $C_c^\infty((\overline{K} \setminus 0) \times \mathbb{R})$ with respect to the norm

$$\|w; H_\beta^s(Q)\| = \left(\sum_{|\alpha| \le s} \int_K \int_{\mathbb{R}} |x|^{2(\beta - s + |\alpha|)} |D_{x,t}^\alpha w(x,t)|^2 \, dx \, dt \right)^{1/2}. \tag{5.11}$$

The norm in $H_\beta^s(Q;q)$ is given by (3.2) with Q in place of K. Let $V_\beta^s(Q,\gamma)$ with $\gamma > 0$ stand for the space with the norm

$$\|w; V_\beta^s(Q;\gamma)\| = \|w^\gamma; H_\beta^s(Q;\gamma)\|, \tag{5.12}$$

where $w^\gamma(x,t) = \exp(-\gamma t) w(x,t)$. It can be checked (see [5]) that the norm $\|w; W_\beta^s(Q;\gamma)\|$ is equivalent to each of the following two norms:

$$\left(\int \|\hat{w}(\cdot,\tau); H_\beta^s(K;p)\|^2 \, d\sigma \right)^{1/2}, \quad \left(\int p^{-n-2(\beta-s)} \|W(\cdot,\tau); H_\beta^s(K;1)\|^2 \, d\sigma \right)^{1/2}, \tag{5.13}$$

where $\hat{w}$ is the Fourier transform of w and $W(\eta,\tau) = \hat{w}(p^{-1}\eta,\tau)$ (we mean that the corresponding constants in the equivalence relations do not depend on $\gamma > 0$).

Consider problem (2.1) with $f \in V_0^0(Q;\gamma)$, $\gamma > 0$. Let $\hat{u}(\cdot,\tau)$ be a strong solution of problem (2.5) with right-hand side $\hat{f}(\cdot,\tau) = F_{t\to\tau} f(\cdot,t)$. The function u defined by $u(x,t) = F_{\tau\to t}^{-1} \hat{u}(x,\tau)$ is called a strong solution of problem (2.1). Theorem 2.5 leads to the following.

Theorem 5.2. *For any $f \in V_0^0(Q;\gamma)$ with $\gamma > 0$ there exists a unique strong solution u of problem* (2.1). *The estimate*

$$\gamma \|u; V_0^1(Q;\gamma)\| \le c\|f; V_0^0(Q;\gamma)\|$$

is valid with constant c independent of $\gamma > 0$.

We fix a function $\chi \in C_c^\infty(\overline{K})$ equal to 1 near the vertex of the cone K and introduce the operator

$$(Xu)(x,t) = F_{\tau\to t}^{-1} \chi(px) F_{t'\to\tau} u(x,t').$$

We also set

$$(\Lambda u)(x,t) = F_{\tau\to t}^{-1} p F_{t'\to\tau} u(x,t').$$

For $\beta \in \mathbb{R}$ and $\gamma > 0$, let $\mathcal{D}V_\beta(Q;\gamma)$ be the space with the norm

$$\|u; \mathcal{D}V_\beta(Q;\gamma)\| = (\|Xu, V_\beta^2(Q;\gamma)\|^2 + \gamma^2 \|u; V_\beta^1(Q;\gamma)\|^2)^{1/2}.$$

If $\beta \leq 1/2$, then the space $\mathcal{R}V_\beta(Q;\gamma)$ is endowed with the norm

$$\|f;\mathcal{R}V_\beta(Q;\gamma)\| = \left(\|f;V^0_\beta(Q;\gamma)\|^2 + \gamma^{-2}\|\Lambda^{1-\beta} f; V^0_0(Q;\gamma)\|^2\right)^{1/2}.$$

For $1/2 \leq \beta \leq 1$ we put

$$\|f;\mathcal{R}V_\beta(Q;\gamma)\| = \left(\|f;V^0_\beta(Q;\gamma)\|^2 + \gamma^{-4\beta}\|\Lambda^{\beta} f; V^0_0(Q;\gamma)\|^2\right)^{1/2}.$$

Clearly, $\mathcal{R}V_\beta(Q;\gamma) \subset V^0_0(Q;\gamma)$.

Applying Theorem 4.6, we obtain the following result.

Theorem 5.3. *Assume that $n \geq 3$ and $f \in \mathcal{R}V_\beta(Q;\gamma)$. If $\beta_1 < \beta \leq 1$, then the strong solution u of problem* (2.1) *belongs to $\mathcal{D}V_\beta(Q;\gamma)$. We have*

$$\|u;\mathcal{D}V_\beta(Q;\gamma)\| \leq c\|f;\mathcal{R}V_\beta(Q;\gamma)\| \tag{5.14}$$

with constant c independent of $\gamma > 0$. In the case where $\beta_{q+1} < \beta < \beta_q$, the strong solution belongs to $\mathcal{D}V_\beta(Q;\gamma)$ if and only if

$$(\hat{f}(\cdot,\tau), w^{(k,j)}_\mu(\cdot,\bar{\tau})) = 0$$

for all $\lambda_\mu \in S_\beta$ and almost all $\tau = \sigma - i\gamma$ with $\sigma \in \mathbb{R}$. Under these conditions, estimate (5.14) *remains valid.*

5.3. The asymptotics of solutions to the problem in a cylinder. To apply Theorem 5.1 to the problem in a cylinder, we must pass to the variables $x = \eta/p$. (The further analysis of the asymptotics is similar to that given in the theory of elliptic boundary-value problems; see [23, 7]. We discuss only a simple situation.) Suppose that $\beta, \beta' \notin \{\beta_q\}$ and that the interval (β', β) contains an element of the sequence $\{\beta_q\}$. Also, assume that $\beta - \beta' < 1$. Let $F \in RH_{\beta'}(K;1)$, and let U satisfy the equation $A_\beta(\theta)U = F$. We shall describe the contribution to the asymptotics for U of the eigenvalues λ_ν located in the strip

$$\beta' - 2 + n/2 < \Im\lambda < \beta - 2 + n/2.$$

Let $S(\beta', \beta)$ be the set of all eigenvalues in this strip. Theorem 5.1 shows that

$$U(\eta) = \chi(\eta) \sum_{\lambda_\nu \in S(\beta',\beta)} d^{(k,j)}_\nu u^{(k,j)}_\nu(\eta) + V(\eta), \tag{5.15}$$

where

$$u^{(k,j)}_\nu(\eta) = |\eta|^{i\lambda\nu} \sum_{q=0}^{k} \frac{1}{q!}(i\log|\eta|)^q \varphi^{(k-q,j)}_\nu(\eta/|\eta|),$$

and $\|V; DH'_\beta(K;1)\| \leq c\|F; RH'_\beta(K;1)\|$. Returning to the variables $x = \eta/p$, we set $u(x) = U(px)$, $f(x) = p^2F(px)$ and obtain

$$\begin{aligned} L(D_x,\tau)u(x) &= f(x), \quad && x \in K,\\ N(x,D_x)u(x) &= 0, && x \in \partial K \setminus 0. \end{aligned}$$

We rewrite (5.15) in the new variables. We have

$$\begin{aligned}\sum_{k=0}^{\kappa_{j,\nu}-1} & d_\nu^{(k,j)} u_\nu^{(k,j)}(\eta) \\ &= |px|^{i\lambda_\nu} \sum_{k=0}^{\kappa_{j,\nu}-1} d_\nu^{(k,j)} \sum_{q=0}^{k} \frac{1}{q!}(i\log|x| + i\log p)^q \varphi_\nu^{(k-q,j)}(\omega) \\ &= p^{i\lambda_\nu} \sum_{k=0}^{\kappa_{j,\nu}-1} d_\nu^{(k,j)} \sum_{q=0}^{k} \frac{1}{q!}(i\log p)^q u_\nu^{(k-q,j)}(x) \\ &= \sum_{k=0}^{\kappa_{j,\nu}-1} u_\nu^{(k,j)}(x) \Big\{ p^{i\lambda_\nu} \sum_{q=0}^{\kappa_{j,\nu}-k-1} \frac{1}{q!}(i\log p)^q d_\nu^{(k+q,j)} \Big\}.\end{aligned}$$

Let $c_\nu^{(k,j)}(p)$ denote the expression in braces. Then (5.15) takes the form

$$u(x) = \chi_p(x) \sum_{\lambda_\nu \in S(\beta',\beta)} c_\nu^{(k,j)}(p) u_\nu^{(k,j)}(x) + v(x), \tag{5.16}$$

where $\chi_p(x) = \chi(px)$ and $v(x) = V(px)$.

Theorem 5.4. *Let β and β' satisfy the conditions listed at the beginning of this subsection. Suppose $f \in \mathcal{R}V_{\beta'}(Q;\gamma)$ and u is a strong solution of problem* (2.1) *in $\mathcal{D}V_\beta(Q;\gamma)$. Then*

$$u(x,t) = \sum (X\check{c}_\nu^{(k,j)})(x,t) u_\nu^{(k,j)}(x) + v(x,t), \tag{5.17}$$

where

$$\begin{aligned}\check{c}_\nu^{(k,j)}(t) &= F^{-1}_{\tau\to t} c_\nu^{(k,j)}(\tau), \\ c_\nu^{(k,j)}(\tau) &= p^{i\lambda_\nu} \sum_{q=0}^{\kappa_{j,\nu}-k-1} \frac{1}{q!}(i\log p)^q d_\nu^{(k+q,j)}(\tau),\end{aligned}$$

and the functions $d_\nu^{(k+q,j)}$ are given by the formula

$$d_\nu^{(k+q,j)}(\tau) = p^{-2}(\hat{f}(\cdot/p,\tau), w_\nu^{(k,j)}(\cdot,\bar{\tau}))_K \tag{5.18}$$

and satisfy

$$\|\check{d}_\nu^{(k,j)}; H^{2-n/2-\beta'}(\mathbb{R})\| \le c\|f; \mathcal{R}V_{\beta'}(Q;\gamma)\|. \tag{5.19}$$

(*The operator X was defined after Theorem* 5.2.) *The asymptotics* (5.17) *involves the terms corresponding to the eigenvalues λ_ν in the strip $\beta' < \Im\lambda + 2 - n/2 < \beta$. The remainder v in* (5.17) *satisfies the estimate*

$$\|v; \mathcal{D}V_{\beta'}(Q;\gamma)\| \le c\|f; \mathcal{R}V_{\beta'}(Q;\gamma)\|. \tag{5.20}$$

Proof. Relation (5.17) follows from (5.16). Formulas (5.18) are deduced from (5.9). To obtain (5.19), we can combine the estimate

$$|d_\nu^{(k,j)}(\tau)| \le p^{-2}\|\hat{f}(\cdot/p,\tau); RH_{\beta'}(K;1)\|,$$

which is a consequence of (5.18), and the identity

$$\|f; \mathcal{R}V_{\beta'}(Q;\gamma)\|^2 = \int_{\mathbb{R}} p^{-n-2\beta'} \|\hat{f}(\cdot/p,\tau); RH_{\beta'}(K;1)\|^2\, d\sigma. \qquad \square$$

We explain the role of the operator X in the asymptotic formula (5.17). The representations for $c_\nu^{(k,j)}$ and estimate (5.19) (which is sharp in the sense of smoothness) show that $c_\nu^{(k,j)}$ can be highly nonsmooth in time. Therefore, the term $c_\nu^{(k,j)}(t)u_\nu^{(k,j)}(x)$ included in the asymptotics (5.17) instead of $(X\check{c}_\nu^{(k,j)})(x,t)u_\nu^{(k,j)}(x)$, could be less smooth than the solution u itself. The operator X smoothes the extension of $c_\nu^{(k,j)}$ to the interior of the cylinder and allows us to avoid the unnatural situation mentioned above. If the function $c_\nu^{(k,j)}$ is sufficiently smooth, then we can do without X. This will be explained by the example of the wave equation. Phenomena of such kind are well studied in the theory of elliptic problems (see, e.g., [23, 7]).

§6. Example: The wave equation

The above results can be detailed and made more explicit for the wave equation. To show this, we recall some information about the spectrum of the pencil $\mathfrak{A}$.

Assume that $P(D_x) = \Delta_x$ is the Laplacian, $n = 2$, K is an angle in $\mathbb{R}^2$ of opening α, $N(x, D_x) = \partial_\nu$, and $\nu = (\nu_1, \nu_2)$ is the unit normal to ∂K. The pencil $\mathfrak{A}$ has simple eigenvalues $\lambda_{\pm k} = \mp k(\pi/\alpha)i$ for $k = 1, 2, \dots$, and the eigenvalue $\lambda_0 = 0$ has multiplicity 2. For $k > 0$, with $\lambda_{\pm k}$ we associate the eigenvectors

$$\Phi_k(\omega) = \phi_k^{(0,1)}(\omega) := (k\pi)^{-1/2}\cos(k\pi\omega/\alpha). \tag{6.1}$$

With the eigenvalue $\lambda_0 = 0$ we associate the eigenvector $\Phi_0(\omega) = \phi_0^{(0,1)}(\omega) := \alpha^{-1/2}$ and the generalized eigenvector $\Phi_{01} = \phi_0^{(1,1)} := 0$.

Now we assume that $n > 2$ and consider $P(D_x) = \Delta_x$ and $N(x, D_x) = \partial_\nu$, where ν is the unit normal to $\partial K \times R^d$. In this case

$$\mathfrak{A}(\lambda) = \{(i\lambda)^2 + (n-2)i\lambda - \delta, \partial_{\nu'}\} : H^2(\Omega) \to L_2(\Omega) \oplus H^{1/2}(\partial\Omega),$$

where δ is the Laplacian on S^{n-1} and ν' is the unit normal to $\partial\Omega$. The spectrum of the pencil $\mathfrak{A}$ consists of the normal eigenvalues

$$\lambda_{\pm k} = \frac{i}{2}\{(n-2) \mp ((n-2)^2 + 4\mu_k)^{1/2}\}, \quad k = 0, 1, \dots,$$

where $\{\mu_k\}$ $(0 = \mu_0 < \mu_1 \le \cdots)$ is the sequence of all eigenvalues of the operator $\{\delta, \partial_{\nu'}\}$ counted with their multiplicities. With the eigenvalues $\lambda_{\pm k}$ we associate the eigenvectors $\Phi_k = \phi_k^{(0,1)}$ chosen so as to have

$$\sqrt{(n-2)^2 + 4\mu_j}(\Phi_j, \Phi_k)_\Omega = \delta_{jk}.$$

There are no generalized eigenvectors. If $n = 3$, then the strip $0 \le \Im\lambda \le 1/2$ contains only one eigenvalue 0, and any eigenvector is a constant.

For the wave equation, the formal series (4.13) and (4.14) converge, and their sums can be found explicitly (see [8, 9]). In the case where $n > 2$ we shall omit the superscripts in (4.13) and (4.14). We have

$$\begin{aligned} U_k(x,\tau) &= \Gamma(1+\nu_k)2^{\nu_k}(ir\sqrt{\tau^2})^{-\nu_k} I_{\nu_k}(ir\sqrt{\tau^2}) r^{i\lambda_{+k}}\Phi_k(\omega), \\ V_k(x,\tau) &= (2^{1-\nu_k}/\Gamma(\nu_k))(ir\sqrt{\tau^2})^{\nu_k} K_{\nu_k}(ir\sqrt{\tau^2}) r^{i\lambda_{-k}}\Phi_k(\omega) \end{aligned} \tag{6.2}$$

with $\nu_k = (\sqrt{(n-2)^2 + 4\mu_k})/2$, where the I_ν and K_ν are modified Bessel functions of the first and the third kind, respectively.

For $n = 2$, there are two additional solutions of the homogeneous problem (4.12); these correspond to the eigenvalue $\lambda_0 = 0$:

$$U_0^{(0,1)}(x,\tau) = \alpha^{-1/2} I_0(ir\sqrt{\tau^2}), \quad V_0^{(1,1)}(x,\tau) = \alpha^{-1/2} K_0(ir\sqrt{\tau^2}), \tag{6.3}$$

where α is the opening of the angle $K \subset R^2$, as before.

Relations (6.2) and (6.3) show that the solutions V_k have power rate of growth near the vertex of the cone and decay exponentially at infinity. The solutions U_k are of class H^1 in a neighborhood of the vertex and grow exponentially at infinity. If $n = 2$, then $V_0^{(1,1)}$ is of logarithmic growth near the vertex and decays rapidly at infinity, while $U_0^{(0,1)}$ grows exponentially at infinity and has a finite limit at the vertex. (Clearly, the V_k are precisely the functions constructed in Proposition 4.2.)

Now we specify Theorems 4.6 and 4.7 for the wave equation. We consider $L(D_x, \tau) = -\Delta - \tau^2$ and $N(x, D_x) = \partial_\nu$. The spaces $RH_\beta(K;p)$ and $DH_\beta(K;p)$ are defined by (4.1), (4.2), and (4.3) (formula (4.3) is valid for any $\beta \le 1$; for the wave equation, we do not need to distinguish the case of $1/2 \le \beta \le 1$). In what follows we denote by $A_\beta(\tau)$ the (closed) operator of problem (2.13) with domain $D(A_\beta(\tau)) \subset DH_\beta(K;p)$.

Theorem 6.1. *Assume that $n \ge 3$. If $1 \ge \beta > 1-(n-2)/2$, then for any $F \in RH_\beta(K;p)$ equation* (4.22) *has a solution. For $\beta \in (\Im\lambda_{k+1} + 2 - n/2, \Im\lambda_k + 2 - n/2)$, equation* (4.22) *is solvable if and only if $(F, V_j(\cdot, \bar\tau)) = 0$, $j = 0, 1, \dots, k$. The solution U is unique and satisfies* (4.23).

Theorem 6.2. *Assume that $n = 2$. If $1 > \beta > \max\{0, 1 - \pi/\alpha\}$, then for any $F \in RH_\beta(K;p)$ equation* (4.22) *has a solution. For the other β, equation* (4.22) *is solvable if the following conditions are fulfilled.*

For $\alpha > \pi$:

(i) $\beta \in (0, 1 - \pi/\alpha)$ *and* $(F, V_1(\cdot, \bar\tau)) = 0$;

(ii) $\beta \in (1 - 2\pi/\alpha, 0]$ *and* $(F, V_{01}(\cdot, \bar\tau)) = 0$, $(F, V_1(\cdot, \bar\tau)) = 0$;

(iii) $\beta \in (1 - (k+1)\pi/\alpha, 1 - k\pi/\alpha)$ *and F is orthogonal to $V_{01}, V_1, \dots, V_k$, where* $k = 2, 3, \dots$.

For $\alpha < \pi$:

(i) $\beta \in (1 - \pi/\alpha, 0]$ *and* $(F, V_{01}(\cdot, \bar\tau)) = 0$;

(ii) $\beta \in (1 - (k+1)\pi/\alpha, 1 - k\pi/\alpha)$ *and F is orthogonal to $V_{01}, V_1, \dots, V_k$, where* $k = 1, 2, \dots$.

The solution U is unique and satisfies (4.23).

We turn to Theorem 5.4. We shall present full asymptotic expansions for the strong solutions. Moreover, we shall show how to get rid of the operator X occurring in (5.18). We deal with the mixed Cauchy–Neumann problem in a semicylinder and restrict ourselves to the case where $d = 0$, $n \ge 3$.

For an open cone K in $\mathbb{R}^n_x$, we consider the problem

$$\begin{cases} (\partial_t^2 - \Delta_x)u(x,t) = 0, \quad (x,t) \in K \times (0, +\infty), \\ \partial_\nu u | \partial K \times (0, +\infty) = 0, \\ u(x,0) = \phi(x), \quad u_t'(x,0) = \psi(x), \end{cases} \tag{6.4}$$

where $\phi, \psi \in C_0^\infty(K)$.

In order to obtain explicit formulas for the coefficients in the asymptotic expansions of solutions, we calculate the inverse Fourier transform $\mathbb{W}_k(x,t) = \mathcal{F}^{-1}_{\tau \to t} V_k(x, \tau)$.

It is known (see, e.g., [24]) that

$$2^{2\mu}\Gamma(2\mu + 1/2)(p/b)^{-2\mu} K_{2\nu}(bp) = \int_{-\infty}^{+\infty} \exp(-pt) P(t)\, dt$$

with $\Re\mu > -1/4$, where

$$P(t) = \begin{cases} 0 & \text{if } t < b, \\ \pi^{1/2}(t^2 - b^2)^{(4\mu-1)/2} F(\mu - \nu, \mu + \nu, 2\mu + 1/2, 1 - t^2/b^2) & \text{if } t > b, \end{cases}$$

and F is the hypergeometric function. Assume that $b = r$, $p = i\tau$, m is a positive integer, $\mu = [\nu] - \nu + m$, and $N = [\nu] + m$; we put $\nu = \nu_k = (\sqrt{(n-2)^2 + 4\mu_k})/2$. Then

$$\begin{aligned}(i\tau)^\nu K_\nu(ir\tau) &= r^{\nu-N}(i\tau)^N\left(\frac{i\tau}{r}\right)^{-\mu} K_\nu(ir\tau)\\ &= \frac{2^{-\mu}}{\Gamma(\mu+1/2)} r^{\nu-N}\mathcal{F}_{t\to\tau}\left(\frac{d}{dt}\right)^N \mathcal{T}_N(r,t,\mu,\nu),\end{aligned}$$

where

$$\begin{aligned}&\mathcal{T}_N(r,t,\mu,\nu)\\ &\quad= \begin{cases} 0 & \text{if } t < r,\\ \pi^{1/2}(t^2-r^2)^{(2\mu-1)/2}F((\mu-\nu)/2, (\mu+\nu)/2, \mu+1/2, 1-t^2/r^2) & \text{if } t > r,\end{cases}\end{aligned} \tag{6.5}$$

and the Fourier transformation and differentiation are understood in the sense of distributions. Thus, we have

$$\mathbb{W}_k(x,t) = \left(\frac{d}{dt}\right)^N \mathcal{P}_{N,k}(x,t) \tag{6.6}$$

with

$$\mathcal{P}_{N,k}(x,t) = \frac{2^{1-N}}{\Gamma(\nu)\Gamma(\mu+1/2)} r^{\nu-\mu+i\lambda_{-k}}\Phi_k(\omega)\mathcal{T}_N(r,t,\mu,\nu), \tag{6.7}$$

where $r = |x|$; in fact, the right-hand side of (6.6) is independent of the choice of m (recall that $N = [\nu] + m$).

In what follows we shall also need the expression

$$U_k(x,\tau) = \Gamma(1+\nu_k) r^{i\lambda_k}\Phi_k(\omega)\sum_{m=0}^{\infty}\frac{(ir\tau)^{2m}}{2^{2m}m!\Gamma(m+\nu_k+1)} \tag{6.8}$$

for the series $U_k(x,\tau)$, which can easily be obtained from the well-known expansion for the Bessel function.

Let $\{\beta_k\}$ be the sequence of all numbers $1 > \beta_1 > \beta_2 > \cdots$ such that every line $\Im\lambda = \beta_k - 2 + n/2$ contains an eigenvalue of the pencil $\mathfrak{A}$.

Theorem 6.3. *Suppose $\beta \in (\beta_{k+1}, \beta_k)$, $\gamma > 0$, $\nu_j = (\sqrt{(n-2)^2 + 4\mu_j})/2$, and $N_j = [\nu_j] + m$, where m is an integer, $m \geq 4$. To be specific, we assume that N_j is even, $N_j = 2l_j$. Set*

$$\check{c}_j(t) = \int_K \Delta^{l_j}\psi(y)\mathcal{P}_{N_j,j}(y,t)\,dy + \int_K \Delta^{l_j}\phi(y)\partial_t\mathcal{P}_{N_j,j}(y,t)\,dy \tag{6.9}$$

with $\mathcal{P}_{N,k}$ given by (6.7). Then for the strong solution u of problem (6.4) we have

$$u(x,t) = \chi(r)\sum_{S_\beta}\Gamma(1+\nu_j)\left\{\sum_{m=0}^{L_j}\frac{(r\partial_t)^{2m}\check{c}_j(t)}{2^{2m}m!\Gamma(m+\nu_j+1)}\right\}\Phi_j(\omega)r^{i\lambda_j} + \rho(x,t), \tag{6.10}$$

where χ is a cut-off function equal to 1 near the origin, the L_j are sufficiently large integers, and $\sum_{S_\beta}$ means the sum of all terms corresponding to the eigenvalues of $\mathfrak{A}$ in the strip $\beta - 2 + n/2 < \Im\lambda \leq (n-2)/2$. The remainder ρ satisfies the estimate

$$\|\rho; DV_\beta(K\times\mathbb{R},\gamma)\| \leq c(\gamma).$$

Proof. Suppose w has the following properties:

1) $w \in C_0^\infty(K \times \mathbb{R})$;

2) $w(x,0) = \phi(x)$, $w_t'(x,0) = \psi(x)$;

3) $((\partial_t^2 - \Delta_x)w)_+ = \theta_+(\partial_t^2 - \Delta_x)w \in C_0^\infty(K \times \mathbb{R})$, where θ_+ is the indicator of the set $\{t : t \geq 0\}$.

Such a function w can be constructed if we observe that conditions 2) and 3) are equivalent to the relations

$$\partial_t^{2n+1} w(x,0) = \Delta_x^n \psi, \;\; \partial_t^{2n} w(x,0) = \Delta_x^n \phi, \quad n = 0,1,2,\dots,$$

and use Borel's theorem (see, e.g., [25, Theorem 1.2.6]). The difference $v = u - w$ satisfies

$$\begin{cases} (\partial_t^2 - \Delta_x)v = -(\partial_t^2 - \Delta_x)w \text{ on } K \times (0,+\infty), \\ \partial_\nu v = 0 \text{ on } \partial K \times (0,+\infty), \\ v|_{t=0} = 0, \quad v_t'|_{t=0} = 0. \end{cases} \tag{6.11}$$

It can be shown that for $t > 0$ the solution of (6.11) coincides with that of the following problem in the cylinder $K \times \mathbb{R}$:

$$\begin{cases} (\partial_t^2 - \Delta_x)v = -((\partial_t^2 - \Delta_x)w)_+ \text{ on } K \times \mathbb{R}, \\ \partial_\nu v = 0 \text{ on } \partial K \times \mathbb{R} = 0. \end{cases} \tag{6.12}$$

Since w vanishes for small $|x|$, the functions u and v coincide near the vertex of K. We put $g = -(\partial_t^2 - \Delta_x)w$ and $g_+ = \theta_+ g$. Theorem 5.4 implies that

$$v(x,t) = \sum_{j \in J} r^{i\lambda_j} \mathcal{U}_j^{L_j}(r\partial_t, \omega)(X\check{c}_j)(x,t) + \check{h}(x,t), \tag{6.13}$$

where

$$\check{c}_j(t) = \mathcal{F}^{-1}_{\tau \to t}(\widehat{g_+}(\cdot,\tau), V_j(\cdot,\bar{\tau}))_{L_2(K)},$$

the operator X is defined after Theorem 5.2, and $r^{i\lambda_j}\mathcal{U}_j^{L_j}(r\partial_t,\omega)$ is the L_jth partial sum of the series (6.8) with ∂_t and j in place of $i\tau$ and k. The remainder $\check{h}$ satisfies (5.20) with f replaced by g_+.

Now we verify (6.9). We have

$$\check{c}_j(t) = \int_K dy \int_{-\infty}^{+\infty} g_+(y,s)\mathbb{W}_j(y,t-s)\,ds.$$

Using (6.7) and integrating by parts twice, we obtain

$$\begin{aligned} \check{c}_j(t) &= (-1)^{N_j} \int_K dy \int_0^{+\infty} \partial_s^{N_j} g(y,s)\mathcal{P}_{N_j,j}(y,t-s)\,ds \\ &= (-1)^{N_j}\Bigg(\int_K dy \int_0^{+\infty} \partial_s^{N_j} w(y,s)\Delta_y \mathcal{P}_{N_j,j}(y,t-s)\,ds \\ &\qquad + \int_K dy \Big\{ \partial_s^{N_j+1} w(y,0)\mathcal{P}_{N_j,j}(y,t) + \partial_s^{N_j} w(y,0)\partial_s \mathcal{P}_{N_j,j}(y,t) \\ &\qquad\qquad - \int_0^{+\infty} \partial_s^{N_j} w(y,s)\partial_s^2 \mathcal{P}_{N_j,j}(y,t-s)\,ds \Big\}\Bigg). \end{aligned} \tag{6.14}$$

Since the integrals over $K \times \mathbb{R}_+$ cancel, we have

$$\check{c}_j(t) = (-1)^{N_j}\left(\int_K \partial_t^{N_j+1} w(y,0)\mathcal{P}_{N_j,j}(y,t)\,dy + \int_K \partial_t^{N_j} w(y,0)\partial_t \mathcal{P}_{N_j,j}(y,t)\,dy \right). \tag{6.15}$$

Recalling that $N_j = 2l_j$, we arrive at (6.9).

It remains to show how to eliminate the operator X from (6.13). We have

$$(X\check{c}_j)(x,t) - \chi(r)\check{c}_j(t) = \int_{\Im\tau=-\gamma} \exp(it\tau)(\chi(|\tau|r) - \chi(r))c_j(\tau)\,d\tau \tag{6.16}$$

with

$$c_j(\tau) = (\widehat{g_+}(\cdot,\tau), V_j(\cdot,\bar{\tau}))_{L_2(K)}.$$

Since $g_+ \in C_0^\infty(K\times\mathbb{R})$, we can use the expression for V_j and the asymptotics of the Bessel function to obtain

$$\left|\left(\frac{d}{d\tau}\right)^k c_j(\tau)\right| \le c(\gamma,k,N)|\tau|^{-N}, \quad k,N \ge 0. \tag{6.17}$$

From (6.16) and (6.17) it follows that the difference

$$\kappa(x,t) = \sum_{j\in J} r^{i\lambda_j}\mathcal{U}_j^{L_j}(r\partial_t,\omega)(X\check{c}_j(x,t)) - \chi(r)\sum_{j\in J} r^{i\lambda_j}\mathcal{U}_J^{L_j}(r\partial_t,\omega)\check{c}_j(t)$$

belongs to $V^s_{\beta'}(K\times\mathbb{R};\gamma)$ for any $s\in\mathbb{N}_0$ and any $\beta'\in\mathbb{R}$. In particular, the norm $\|\kappa; DV_\beta(K\times\mathbb{R},\gamma)\|$ is finite. Setting $\rho(x,t) = \kappa(x,t) + \check{h}(x,t)$, we obtain the desired result. $\square$

§7. A bounded domain with a conical point

Let G be a bounded domain in $\mathbb{R}^n$ with boundary ∂G that is smooth outside a point O. We assume that O is the origin and that in a neighborhood of O the domain G coincides with an open cone K that cuts out a domain Ω from the sphere S^{n-1}; next, it is assumed that Ω has smooth boundary. We consider the problem

$$\begin{aligned} L(D_x,D_t)u(x,t) &= f(x,t), && (x,t)\in G\times\mathbb{R},\\ N(x,D_x)u(x,t) &= 0, && (x,t)\in\partial G\times\mathbb{R}, \end{aligned} \tag{7.1}$$

where the operators $L(D_x,D_t) = P(D_x) - D_t^2$ and $N(x,D_x)$ are the same as in (2.1).

Proposition 7.1. *Suppose $v\in C^\infty(\bar{G})$, $N(x,D_x)v = 0$ on $\partial G = 0$, and the operator $P(D_x)$ satisfies condition* a) *in* 2.1. *Then*

$$\gamma^2\int_G (|\tau|^2|v(x)|^2 + |\nabla v(x)|^2)\,dx \le c\int_G |L(D_x,\sigma - i\gamma)v(x)|^2\,dx \tag{7.2}$$

with constant c independent of $\tau = \sigma - i\gamma$, where $\sigma\in\mathbb{R}$ and $\gamma > 0$.

If $P(D_x)$ is the Lamé operator (*and $n = 2,3$*), *then estimate* (7.2) *remains valid for $\gamma\ge\gamma_0$ with sufficiently large γ_0, and the constant c depends only on γ_0.*

Proof. Suppose that $u\in S(\mathbb{R}^{n+1}_{x,t})$ and $N(x,D_x)u = 0$ on $\partial G\times\mathbb{R}$. As in the proof of Proposition 2.1, we obtain the inequality

$$\|u_t(\cdot,t);T\|^2 + (P(D_x)u(\cdot,t),u(\cdot,t))_T = 2\Re\int_T\int_{-\infty}^t \langle f,u_t\rangle\,dx\,dt, \tag{7.3}$$

where $f(x,t) = (\partial_t^2 + P(D_x))u(x,t)$. If the operator $P(D_x)$ satisfies condition a), then

$$(P(D_x)u(\cdot,t),u(\cdot,t))_T = a(u(\cdot,t),u(\cdot,t)) \ge c_1\|\nabla u;T\|^2, \tag{7.4}$$

and we can proceed as in the proof of Proposition 2.1. For the Lamé operator, estimate (7.4) fails. We use the Korn inequality

$$(P(D_x)u,u)_T = a(u,u) \ge c\|\nabla u;T\|^2 - c_1\|u;T\|^2. \tag{7.5}$$

By (7.5) and (7.3), we have

$$\gamma^2 \int_T dy \int_{-\infty}^{+\infty} d\sigma |\hat{\psi}(\sigma - i\gamma)|^2 (|\tau|^2 |v(y)|^2 + |\nabla v(y)|^2 - c_3 |v(y)|^2)$$
$$\leq c \int_T dy \int_{-\infty}^{+\infty} d\sigma |\hat{\psi}(\sigma - i\gamma)|^2 |L(D_y, \zeta, \sigma - i\gamma) |v(y)|^2$$

for any $\psi \in \exp(-\gamma t) S(\mathbb{R}) \cap S(\mathbb{R})$. This implies (7.2) for sufficiently large γ. □

We introduce the operator $u \mapsto L(D_x, \tau)u$ on $L_2(G)$ with domain

$$\{u \in C^\infty(\bar{G} \setminus O) \cap H^1(G) : N(x, D_x)u|\partial G = 0, L(D_x, \tau)u \in L_2(G)\}.$$

The closure of this operator will be denoted by $A(\tau)$. A solution of the equation $A(\tau)u = f \in L_2(G)$ will be called a strong solution of the problem

$$\begin{aligned} L(D_x, \tau)u(x) &= f, \quad x \in G, \\ N(x, D_x)u(x) &= 0, \quad x \in \partial G. \end{aligned} \tag{7.6}$$

As in Subsection 2.4, we obtain the following result.

Theorem 7.2. *Suppose the operator $P(D_x)$ satisfies condition* a) *in* 2.1. *For any $f \in L_2(G)$ and any $\tau = \sigma - i\gamma$ with $\sigma \in \mathbb{R}$ and $\gamma > 0$, there exists a unique strong solution u of problem* (7.6). *We have*

$$\gamma^2 (|\tau|^2 \|u; L_2(G)\|^2 + \|\nabla u; L_2(G)\|^2) \leq c \|f; L_2(G)\|^2$$

with constant c independent of σ and γ.

If $P(D_x)$ is the Lamé operator, then the same is true provided $\gamma \geq \gamma_0 > 0$ with sufficiently large γ_0.

We introduce the spaces $DH_\beta(G; |\tau|)$ and $RH_\beta(G; |\tau|)$ endowed with the norms (4.1)–(4.3), where K is replaced by G and $p = |\tau|$. As before, for the wave equation we define the space $RH_\beta(G; |\tau|)$ by (4.3) with $\beta \leq 1$.

Proposition 7.3. *Suppose that $\beta \leq 1$, the line $\Im \lambda = \beta - 2 + n/2$ contains no eigenvalues of the pencil $\mathfrak{A}$, and $\gamma \geq \gamma_0$ with large $\gamma_0 > 0$. Then for any function $u \in C_c^\infty(\bar{G} \setminus O)$ such that $N(x, D_x)u|\partial G = 0$ we have the estimate*

$$\|u; DH_\beta(G; |\tau|)\| \leq c \|L(D_x, \tau)u; RH_\beta(G; |\tau|)\| \tag{7.7}$$

with constant c independent of σ and γ.

Proof. We outline the proof for the wave equation. (In the general case, the argument given below fails, because ψu with a cut-off function ψ does not satisfy the boundary condition $N(x, D_x)(\psi u)|\partial G = 0$, so that we cannot apply inequality (3.26) directly. To obtain the required estimate, one must argue as in the proof of Proposition 3.1.)

For instance, assume that $n > 2$ and ψ is a cut-off function equal to 1 near the point O and supported in the neighborhood of O where G coincides with K. Also, we assume that ψ depends only on $|x|$. We have

$$\begin{aligned} L(D_x, \tau)(\psi u) &= \psi L(D_x, \tau)u + [L(D_x, \tau), \psi]u, \quad && x \in G, \\ \partial_\nu (\psi u) &= 0, && x \in \partial G. \end{aligned} \tag{7.8}$$

By (3.26),

$$\begin{aligned} &\|\psi u; DH_\beta(K; |\tau|)\| \\ &\quad \leq c \left(\|\psi L(D_x, \tau)u; RH_\beta(K; |\tau|)\| + \|[L(D_x, \tau), \psi]u; RH_\beta(K; |\tau|)\| \right). \end{aligned} \tag{7.9}$$

The definition of the norm on $RH_\beta(K; |\tau|)$ implies that

$$\|[L, \psi]u; RH_\beta(K; |\tau|)\| \leq c(\|[L, \psi]u; H^0_\beta(K)\| + |\tau|^{1-\beta} \gamma^{-1} \|[L, \psi]u; L_2(K)\|).$$

Estimate (7.2) yields

$$\|[L,\psi]u; L_2(K)\| \le \|v; H_0^1(G;|\tau|)\| \le \|L(D_x,\tau)u; L_2(G)\|;$$

moreover, $\|[L,\psi]u; H_\beta^0(K)\| \le c\|u; H_\beta^1(G;|\tau|)\|$. This leads to the inequality

$$\|\psi u; DH_\beta(G;|\tau|)\| \le c(\|u; H_\beta^1(G;|\tau|)\| + \|L(D_x,\tau)u; RH_\beta(G;|\tau|)\|).$$

We use (7.2) once again to obtain

$$\|(1-\psi)u; DH_\beta(G;|\tau|)\| \le c\|u; H_0^1(G;|\tau|)\| \le c\|Lu; RH_\beta(G;|\tau|)\|. \tag{7.10}$$

Adding up (7.9) and (7.10), we arrive at (7.7) with sufficiently large γ. □

The method used in §§4 and 5 can easily be adapted to problem (7.6). In particular, the role of functions $w_\mu^{(k,j)}$ constructed in Proposition 4.2 can be played by the functions $W_\mu^{(k,j)}$ that are solutions of the homogeneous problem (7.6) with asymptotics $V_{\mu,T}^{(k,j)}(x,\tau)$ near O. The construction of $W_\mu^{(k,j)}$ is essentially the same as that of $w_\mu^{(k,j)}$.

Let $\hat u(\cdot,\tau)$ be a strong solution of problem (7.6) with the right-hand side $\hat f(\cdot,\tau) = F_{t\to\tau} f(\cdot,t)$. The function $(x,t)\mapsto u(x,t) = F^{-1}_{\tau\to t}\hat u(x,\tau)$ is called a strong solution of problem (7.1). The spaces $RV_\beta(Q;\gamma)$ and $DV_\beta(Q;\gamma)$ in the cylinder $G\times\mathbb{R}$ are introduced as in Subsection 5.2, with the replacement of K by G in all intermediate formulas. The following analog of Theorem 5.4 is true.

Theorem 7.4. *Let $\gamma \ge \gamma_0$ with sufficiently large γ_0, and let β and β' be the same as in Theorem* 5.4. *If $f \in RV_{\beta'}(Q;\gamma)$, and $u \in DV_\beta(Q;\gamma)$ is a strong solution to problem* (7.1), *then*

$$u(x,t) = \sum (X\check c_\nu^{(k,j)})(x,t) u_\nu^{(k,j)}(x) + v(x,t), \tag{7.11}$$

where

$$\check c_\nu^{(k,j)}(t) = F^{-1}_{\tau\to t} c_\nu^{(k,j)}(\tau),$$
$$c_\nu^{(k,j)}(\tau) = |\tau|^{i\lambda_\nu} \sum_{q=0}^{\kappa_{j,\nu}-k-1} \frac{1}{q!}(i\log|\tau|)^q d_\nu^{(k+q,j)}(\tau),$$

and the functions $d_\nu^{(k+q,j)}$ are given by the formula

$$d_\nu^{(k+q,j)}(\tau) = |\tau|^{-2}(\hat f(\cdot/|\tau|,\tau), W_\nu^{(k,j)}(\cdot,\bar\tau))_T \tag{7.12}$$

and satisfy

$$\|\check d_\nu^{(k,j)}; H^{-n/2+2-\beta'}(\mathbb{R})\| \le c\|f; \mathcal{R}V_{\beta'}(Q;\gamma)\|. \tag{7.13}$$

The operator X is defined by the relation

$$(Xw)(x,t) = F^{-1}_{\tau\to t}\chi(|\tau|x)F_{t'\to\tau}w(t').$$

The asymptotics (7.11) *consists of the terms corresponding to the eigenvalues λ_ν in the strip $\beta' < \Im\lambda + 2 - n/2 < \beta$. The remainder v in* (7.11) *satisfies the estimate*

$$\|v; \mathcal{D}V_{\beta'}(Q;\gamma)\| \le c\|f; \mathcal{R}V_{\beta'}(Q;\gamma)\|.$$

The role played by the operator X in the asymptotics (7.11) was explained after the proof of Theorem 5.4.

References

[1] J.-L. Lions and E. Magenes, *Problèmes aux limites non homogènes et applications*. Vol. 1, Dunod, Paris, 1968. MR0247243 (40:512)

[2] V. A. Kondrat′ev and O. A. Oleĭnik, *Boundary value problems for a system in elasticity theory in unbounded domains. Korn inequalities*, Uspekhi Mat. Nauk **43** (1988), no. 5, 55–98; English transl., Russian Math. Surveys **43** (1988), no. 5, 65–119. MR0971465 (89m:35061)

[3] G. Fichera, *Existence theorems in elasticity. Boundary value problems of elasticity with unilateral constraints*, Handbuch der Physik, Band 6a/2, Springer-Verlag, Berlin–New York, 1972. MR0347187 (49:11907)

[4] V. A. Kozlov and V. G. Maz′ya, *Spectral properties of operator pencils generated by elliptic boundary value problems in a cone*, Funktsional. Anal. i Prilozhen. **22** (1988), no. 2, 38–46; English transl., Funct. Anal. Appl. **22** (1988), no. 2, 114–121. MR0947604 (90a:47125)

[5] B. A. Plamenevskiĭ, *On the Dirichlet problem for the wave equation in a cylinder with edges*, Algebra i Analiz **10** (1998), no. 2, 197–228; English transl., St. Petersburg Math. J. **10** (1999), no. 2, 373–397. MR1629407 (99i:35087a)

[6] A. Yu. Kokotov and B. A. Plamenevskiĭ, *On the Cauchy–Dirichlet problem for hyperbolic systems in a wedge*, Algebra i Analiz **11** (1999), no. 3, 140–195; English transl., St. Petersburg Math. J. **11** (2000), no. 3, 497–534. MR1711368 (2000j:35170)

[7] S. A. Nazarov and B. A. Plamenevskiĭ, *Elliptic problems in domains with piecewise smooth boundaries*, de Gruyter Exp. Math., vol. 13, Walter de Gruyter, Berlin, 1994. MR1283387 (95h:35001)

[8] A. Yu. Kokotov, P. Neittaämaki, and B. A. Plamenevskiĭ, *Problems of diffraction by a cone: asymptotic behavior of the solutions near the vertex*, Zap. Nauchn. Sem. S.-Peterburg. Otdel. Mat. Inst. Steklov. (POMI) **259** (1999), 122–144; English transl., J. Math. Sci. **109** (2002), no. 5, 1894–1910. MR1754360 (2001m:35195)

[9] ———, *The Neumann problem for the wave equation in a cone*, Probl. Mat. Anal., vyp. 20, "Nauchn. Kniga", Novosibirsk, 2000, pp. 71–110; English transl., J. Math. Sci. **102** (2000), no. 5, 4400–4428. MR1807064 (2002k:35180)

[10] V. G. Maz′ya and B. A. Plamenevskiĭ, *The coefficients in the asymptotics of the solutions of elliptic boundary value problems in a cone*, Zap. Nauchn. Sem. Leningrad. Otdel. Mat. Inst. Steklov. (LOMI) **52** (1975), 110–127; English transl., J. Soviet Math. **9** (1978), no. 5, 750–764. MR0407445 (53:11220)

[11] V. A. Kozlov, V. G. Maz′ya, and J. Rossmann, *Elliptic boundary value problems in domains with point singularities*, Math. Surveys Monographs, vol. 52, Amer. Math. Soc., Providence, RI, 1997. MR1469972 (98f:35038)

[12] S. A. Nazarov and B. A. Plamenevskiĭ, *The Neumann problem for selfadjoint elliptic systems in a domain with a piecewise-smooth boundary*, Trudy Leningrad. Mat. Obshch. **1** (1991), 174–211; English transl., Amer. Math. Soc. Transl. Ser. 2, vol. 155, Amer. Math. Soc., Providence, RI, 1993, pp. 169–206. MR1104210 (92e:35065)

[13] G. Eskin, *The wave equation in a wedge with general boundary conditions*, Comm. Partial Differential Equations **17** (1992), no. 1–2, 99–160. MR1151258 (92m:35152)

[14] J. Cheeger and M. Taylor, *On the diffraction of waves by conical singularities*. I, II, Comm. Pure Appl. Math. **35** (1982), 275–331, 487–529. MR0649347 (84h:35091a); MR0657825 (84h:35091b)

[15] M. Uchida, *Microlocal analysis of diffraction by a corner*, Ann. Sci. École Norm. Sup. (4) **25** (1992), no. 1, 47–75. MR1152613 (93b:35004)

[16] P. Gérard and G. Lebeau, *Diffusion d'une onde par un coin*, J. Amer. Math. Soc. **6** (1993), no. 2, 341–424. MR1157289 (93f:35130)

[17] P. Grisvard, *Contrôlabilité exacte des solutions de l'équation des ondes en présence de singularités*, J. Math. Pures Appl. (9) **68** (1989), 215–259. MR1010769 (90i:49045)

[18] V. A. Borovikov, *Diffraction by polygons and polyhedra*, "Nauka", Moscow, 1966. (Russian) MR0218058 (36:1147)

[19] V. B. Poruchikov, *Methods of the dynamic theory of elasticity*, "Nauka", Moscow, 1986. (Russian) MR0846431 (87g:73033)

[20] V. A. Dobrushkin, *Boundary value problems of the dynamic theory of elasticity for wedge-shaped domains*, "Nauka i Tekhnika", Minsk, 1988. (Russian) MR0993072 (91e:73023)

[21] I. I. Mel′nikov, *Singularities of the solution of a mixed problem for second-order hyperbolic equations in domains with a piecewise-smooth boundary*, Uspekhi Mat. Nauk **37** (1982), no. 1, 149–150; English transl., Russian Math. Surveys **37** (1982), no. 1, 168–169. MR0643778 (84j:35106)

[22] Nguyen Manh Hung, *Asymptotics of solutions of the first boundary-value problem for strongly hyperbolic systems near a conical point of the boundary of a domain*, Mat. Sb. **190** (1999), no. 7, 103–126; English transl., Sb. Math. **190** (1999), no. 7, 1035–1058. MR1725214 (2000m:35117)

[23] V. G. Maz′ya and J. Rossmann, *Über die Asymptotik der Lösungen elliptischer Randwertaufgaben in der Umgebung von Kanten*, Math. Nachr. **138** (1988), 27–53. MR0975198 (90a:35079)

[24] A. Erdélyi, W. Magnus, F. Oberhettinger, and F. G. Tricomi, *Higher transcendental functions.* Vols. I, II, McGraw-Hill Book Co., Inc., New York etc., 1953. MR0058756 (15:419i)

[25] L. Hörmander, *The analysis of linear partial differential operators.* 1. *Distribution theory and Fourier analysis*, Grundlehren Math. Wiss., vol. 256, Springer-Verlag, Berlin, 1983. MR0717035 (85g:35002a)

[26] V. G. Maz′ya and B. A. Plamenevskiĭ, *Asymptotic behavior of the fundamental solutions of elliplic boundary value problems in domains with conical points*, Probl. Mat. Anal., vyp. 7, Leningrad. Univ., Leningrad, 1979, pp. 100–145; English transl. in J. Soviet Math. **35** (1986), no. 1. MR0559106 (81e:35042)

Concordia University, Montreal, Canada
E-mail address: kokotov@online.ru

Department of Physics, St. Petersburg State University, Ulyanovskaya 1, Petrodvorets, St. Petersburg 198504, Russia
E-mail address: boris.plamenevskij@pobox.spbu.ru

Received 1/DEC/2003

Translated by B. A. PLAMENEVSKIĬ